Ma 227 **Exam III B Solutions** **4/26/06**

Name: ________________________________ **ID**: _________________________

Lecture Section: ________________________________ **Lecturer**: _________________________

I pledge my honor that I have abided by the Stevens Honor System. ___________________________________

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

Total Score ________

1a [**15 pts**.] Find the work done by the force field

$$\vec{F}(x,y) = (-x-y)\vec{i} + 3xy\vec{j}$$

along the plane path that is the graph of $y = 2x^3 - 1$ from $A = (0,-1)$ to $B = (1,1)$.
Solution: The path may be parametrized as $C : x = t, y = 2t^3 - 1 \quad 0 \le t \le 1$. Then
$\vec{r}(t) = t\vec{i} + (2t^3-1)\vec{j},\ \vec{r}'(t) = \vec{i} + 6t^2\vec{j},\ \vec{F}(t) = (-t-2t^3+1)\vec{i} + 3t(2t^3-1)\vec{j}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(t)\cdot\vec{r}'(t)dt = \int_0^1\left(-t-2t^3+1+18t^3(2t^3-1)\right)dt = \int_0^1(36t^6-20t^3-t+1)dt$$

$$= \frac{36t^7}{7} - \frac{20t^4}{4} - \frac{t^2}{2} + t\Big]_0^1 = \frac{36}{7} - 5 - \frac{1}{2} + 1 = \frac{9}{14}$$

: $\frac{9}{14}$

1b [**15 pts**.] Consider

$$\vec{F} = (5\cos x - e^{2y})\vec{i} + (7\arctan x + 10y)\vec{j} + (e^{\sin 2x} + 4\sin(2z))\vec{k}$$

Find

$$\nabla\left(div(\vec{F})\right) = \nabla\left(\nabla\cdot\vec{F}\right).$$

Solution: $\quad div(\vec{F}) = \frac{\partial}{\partial x}(5\cos x - e^{2y}) + \frac{\partial}{\partial y}(7\arctan x + 10y) + \frac{\partial}{\partial z}(e^{\sin 2x} + 4\sin(2z))$

$$= -5\sin x + 10 + 8\cos(2z)$$

$$\nabla(div(\vec{F})) = -5\cos x\,\vec{i} + 0\vec{j} - 16\sin(2z)\vec{k}$$

2a $\left[20\ \textbf{pts}.\right]$ Find a function $\Phi(x,y,z)$ such that $\nabla\Phi = \vec{F}$, where

$$\vec{F}(x,y,z) = \left(2xyz + e^{2y} + 1\right)\vec{i} + \left(x^2z + 2xe^{2y} + z^2\sin y\right)\vec{j} + \left(x^2y - 2z\cos y + 5\right)\vec{k}$$

Solution: We first note the

$$\nabla\times\left(2xyz + e^{2y} + 1, x^2z + 2xe^{2y} + z^2\sin y, x^2y - 2z\cos y + 5\right) = (0,0,0)$$

so such a Φ exists. This is not required in order to get full credit for the solution.
We have that

$$\Phi_x = 2xyz + e^{2y} + 1 \quad \Phi_y = x^2z + 2xe^{2y} + z^2\sin y \quad \Phi_z = x^2y - 2z\cos y + 5$$

Starting with Φ_x and integrating with respect to x we get

$$\Phi = x^2yz + xe^{2y} + x + h(y,z)$$

Then

$$\Phi_y = x^2z + 2xe^{2y} + \frac{\partial h}{\partial y} = x^2z + 2xe^{2y} + z^2\sin y$$

Thus

$$\frac{\partial h}{\partial y} = z^2\sin y \Rightarrow h(y,z) = -z^2\cos y + g(z)$$

and therefore

$$\Phi = x^2yz + xe^{2y} + x - z^2\cos y + g(z)$$

Hence

$$\Phi_z = x^2y - 2z\cos y + g'(z) = x^2y - 2z\cos y + 5$$

Then

$$g'(z) = 5 \Rightarrow g(z) = 5z + k$$

Finally we have that

$$\Phi = x^2yz + xe^{2y} + x - z^2\cos y + 5z + k$$

Check: SNB $\nabla\left(x^2yz + xe^{2y} + x - z^2\cos y + 5z + k\right) = \begin{matrix} e^{2y} + 2xyz + 1 \\ x^2z + 2e^{2y}x + (\sin y)z^2 \\ yx^2 - 2z\cos y + 5 \end{matrix}$

2b [**20 pts**.] Verify that Green's Theorem is true for the line integral

$$\oint_C -ydx + xdy$$

where C is the circle with center at the origin and radius 4.

Solution: The circle can be parameterized by

$$x = 4\cos t, \quad y = 4\sin t, \quad 0 \le t \le 2\pi$$

$$\oint_C -ydx + xdy = \int_0^{2\pi}(-4\sin t(-4\sin t) + 4\cos t(4\cos t))dt = \int_0^{2\pi} 16(\sin^2 t + \cos^2 t)dt = \int_0^{2\pi} 16dt = 32\pi$$

Also since $P = -y$ and $Q = x$,

$$\iint_R (Q_x - P_y)dA = \int\int(1 - (-1))dA = \int\int 2dA = 2(\text{Area of } x^2 + y^2 = 16) = 2\pi(4)^2 = 32\pi$$

3 a [**10 pts**.] Let S be the portion of $r = \theta^2$ that lies between $z = x^2 + y^2$ and $z = 4$. Use cylindrical coordinates to give a parametrization of S.

Solution: Let $x = \theta^2\cos\theta, y = \theta^2\sin\theta, z = z, \ (\theta^2)^2 \le z \le 4, \ 0 \le \theta \le 2\pi$ or

$$\vec{r}(\theta, z) = \theta^2\cos\theta\vec{i} + \theta^2\sin\theta\vec{j} + z\vec{k} \quad \theta^4 \le z \le 5, \ 0 \le \theta \le 2\pi$$

3 b [**20 pts**.] Give an expression for

$$\iint_S ydS$$

where S is the surface in part 3a. Do *not* evaluate your expression.

Solution: For a surface given by

$$x = x(u,v) \qquad y = y(u,v) \qquad z = z(u,v)$$

that

$$\int\int_S f(x,y,z)ds = \int\int_G f(u,v)|\vec{r}_u \times \vec{r}_v|dudv,$$

where G is the image of the surface S in the u,v -plane. Letting $u = \theta$ and $v = z$, we have

$$\vec{r}(\theta, z) = \theta^2\cos\theta\vec{i} + \theta^2\sin\theta\vec{j} + z\vec{k}$$

and

$$\vec{r}_\theta(\theta, z) = (2\theta\cos\theta - \theta^2\sin\theta)\vec{i} + (2\theta\sin\theta + \theta^2\cos\theta)\vec{j}$$
$$\vec{r}_z(\theta, z) = \vec{k}$$

Thus

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2\theta\cos\theta - \theta^2\sin\theta & 2\theta\sin\theta + \theta^2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2\theta\sin\theta + \theta^2\cos\theta)\vec{i} - (2\theta\cos\theta - \theta^2\sin\theta)\vec{j}$$

so

$$\begin{aligned} |\vec{r}_\theta \times \vec{r}_z| &= \sqrt{(2\theta\sin\theta + \theta^2\cos\theta)^2 + (2\theta\cos\theta - \theta^2\sin\theta)^2} \\ &= \sqrt{4\theta^2\sin^2\theta + 4\theta^3\sin\theta\cos\theta + \theta^4\cos^2\theta + 4\theta^2\cos^2\theta - 4\theta^3\sin\theta\cos\theta + \theta^4\sin^2\theta} \\ &= \sqrt{4\theta^2 + \theta^4} \end{aligned}$$

Thus

$$\iint_S y\,dS = \int_0^{2\pi}\int_{\theta^4}^{5} \theta^2\sin\theta\left(\sqrt{4\theta^2 + \theta^4}\right)dz\,d\theta$$