Ma 227 **Exam III B Solutions** **12/6/11**

Name: ________________________________

Lecture Section: ________ **Recitation Section**: _______________

1a [**15 pts**.] Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F}(x,y,z) = x\vec{i} - y\vec{j} + z^2\vec{k}$ and C is the curve given by $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k},\ 0 \leq t \leq 2\pi$.
Solution:

$$\vec{F}(t) = \cos\vec{i} - \sin t\vec{j} + t^2\vec{k}$$

$$\vec{r}'(t) = -\sin t\vec{i} + \cos t\vec{j} + \vec{k}$$

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \vec{F}(t) \cdot \vec{r}(t)dt \\
&= \int_0^{2\pi} \left(-\sin t\cos t - \sin t\cos t + t^2\right)dt \\
&= \left[\frac{-2}{2}\sin^2 t + \frac{t^3}{3}\right]_0^{2\pi} \\
&= -\sin^2 t + \frac{t^3}{3}\Bigg|_0^{2\pi} = \frac{8\pi^3}{3}
\end{aligned}$$

1b [**15 pts**.] Let $g(x,y,z)$ be a scalar function whose cross partial derivatives are continuous. Calculate $\nabla \times \nabla g$.

Solution:

$$\nabla g = g_x\vec{i} + g_y\vec{j} + g_z\vec{k}$$

$$\nabla \times \nabla g = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ g_x & g_y & g_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ g_x & g_y & g_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ g_x & g_y \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{matrix}$$

$$= g_{zy}\vec{i} + g_{xz}\vec{j} + g_{yx}\vec{k} - g_{xy}\vec{k} - -g_{yz}\vec{i} - g_{zx}\vec{j} = 0$$

2 a [14 **pts**.] Find a function $\phi(x,y,z)$ such that

$$\nabla\phi = \vec{F}(x,y,z) = 4x^3y^3z^2\vec{i} + \left(3x^4y^2z^2 + 2y\right)\vec{j} + \left(2x^4y^3z - 4z\right)\vec{k}$$

Solution:

$$\phi_x = 4x^3y^3z^2$$

$\Rightarrow$

$$\phi = x^4y^3z^2 + g(y,z)$$

Then

$$\phi_y = 3x^4y^2z^4 + g_y = 3x^4y^2z^2 + 2y$$

$$\therefore \; g_y = 2y$$

Thus

$$g(y,z) = y^2 + h(z)$$

and

$$\phi = x^4y^3z^2 + y^2 + h(z)$$

Then

$$\phi_z = 2x^4y^3z + h'(z) = 2x^4y^3z - 4z$$

Hence

$$h'(z) = -4z$$

$$h(z) = -2z^2 + K$$

and

$$\phi = x^4y^3z^2 + y^2 - 2z^2 + K.$$

Any value of K may be chosen since one function is requested.

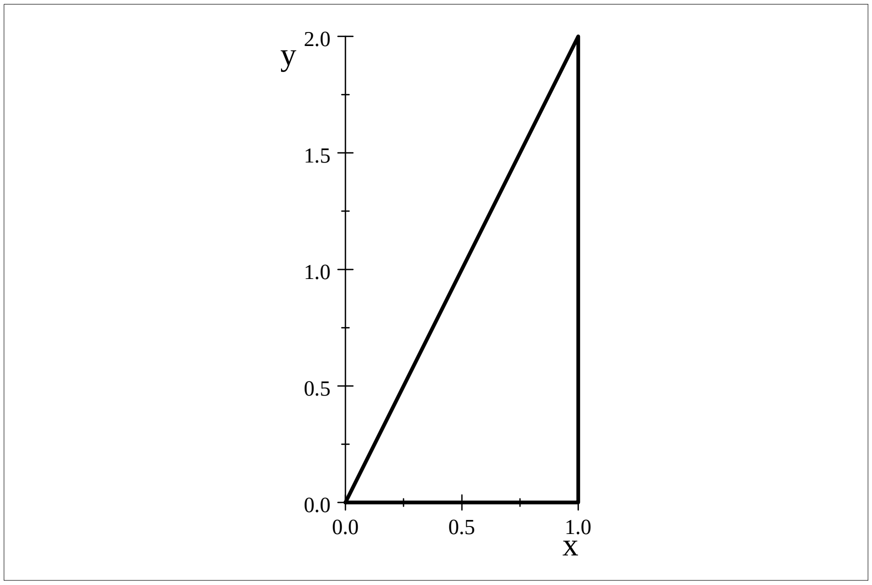
2 b [6 **pts**.] Let Γ be any closed curve in three space. What is the value of $\oint_\Gamma \vec{F} \cdot d\vec{r}$ where $\vec{F}$ is the vector function in 2a. Justify your conclusion.

Solution: Since $\vec{F}$ is derivable from a potential function, then it is a conservative force field, the line integral is independent of path, and $\oint_\Gamma \vec{F} \cdot d\vec{r} = 0$ for any closed curve Γ.

3a $\left[15 \textbf{ pts.} \right]$ Evaluate

$$\oint_C xydx + x^2y^2dy$$

directly without using Green's Theorem, where C is the triangle with vertices $(0,0),(1,0),(1,2)$.
Solution: The triangle C is shown below.
$(0,0,1,0,1,0,1,2,0,0)$



The hypotenuse of the triangle is the line $y = 2x$. We traverse the triangle in the counterclockwise direction starting with the horizontal leg.

$$C_1 : x = t,\ \ y = 0 \ \ 0 \le t \le 1$$
$$C_2 : x = 1,\ y = t \ \ 0 \le t \le 2$$
$$C_3 : x = t,\ y = 2t \ \ t : 1 \to 0$$

$$\begin{aligned}\oint_C xydx + x^2y^3dy &= \int_{C_1} + \int_{C_2} + \int_{C_3} \\ &= 0 + \int_0^2 (1)^2 t^2 dt + \int_1^0 t(2t)dt + \int_1^0 t^2(2t)^2(2)dt \\ &= \frac{t^3}{3}\Big|_0^2 + \frac{2t^3}{3}\Big|_1^0 + 8\frac{t^5}{5}\Big|_1^0 \\ &= \frac{8}{3} - \frac{2}{3} - \frac{8}{5} = \frac{2}{5}\end{aligned}$$

3b [10 **pts**.] Evaluate the line integral in 3a, namely

$$\oint_C xydx + x^2y^2dy$$

using Green's Theorem.

Solution: $P = xy$ and $Q = x^2y^3$. Green's Theorem is

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Thus

$$\oint_C xydx + x^2y^3dy = \iint\limits_{\text{Triangle}} \left(2xy^2 - x\right)dA$$

$$= \int_0^1 \int_0^{2x} \left(2xy^2 - x\right)dydx$$

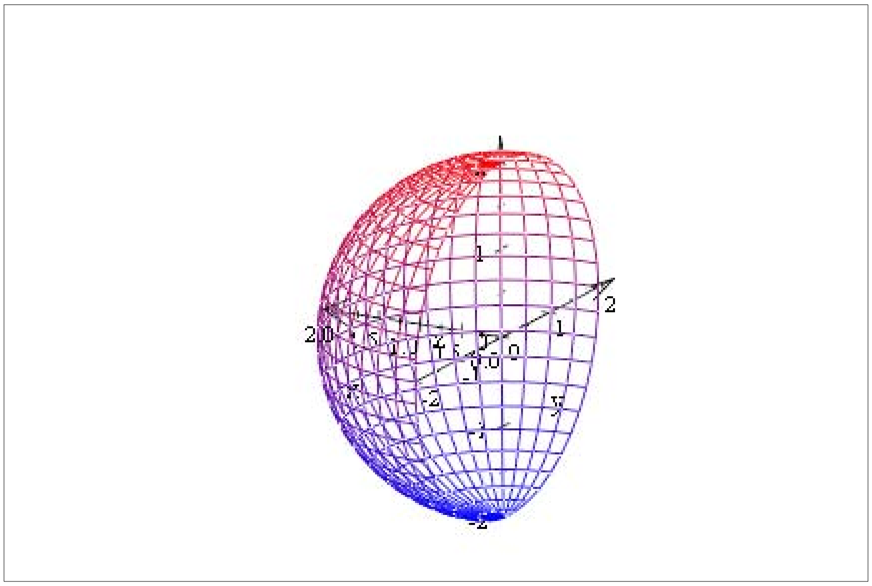
$$= \int_0^1 \left[\frac{2xy^3}{3} - xy \right]_0^{2x} dx$$

$$= \int_0^1 \left(\frac{16x^4}{3} - 2x^2 \right)dx = \frac{16x^5}{15} - \frac{2x^3}{3} \Big|_0^1 = \frac{16}{15} - \frac{10}{15} = \frac{2}{5}$$

4*a* [10 **pts**] Consider the sphere of radius 2 centered at the origin. Let *S* be the half of the sphere forward of the $y - z$ plane (i.e. with $x \geq 0.$) Sketch and parametrize the surface *S*. Be sure to include the range of the parameters.

Solution.

$(2, \theta, \phi)$



Since we are dealing with a hemisphere, we use spherical coordinates to parametrize.

$$x = \rho\cos\theta\sin\phi,\ \ y = \rho\sin\theta\sin\phi,\ \ z = \rho\cos\phi$$

For this hemisphere $\rho = 2$ so

$$x = 2\cos\theta\sin\phi,\ \ y = 2\sin\theta\sin\phi,\ \ z = 2\cos\phi$$

where $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$, and $0 \leq \phi \leq \pi$.

4*b* [15 **pts.**] Give an iterated double integral equal to

$$\iint_S xdS$$

where *S* is the surface in part 3a. Do *not* evaluate your expression.

Solution:

$$\vec{r}(\theta,\phi) = 2\cos\theta\sin\phi\vec{i} + 2\sin\theta\sin\phi\vec{j} + 2\cos\phi\vec{k}$$

$$\iint_S x\,dS = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} (2\cos\theta\sin\phi)\left|\vec{r}_\theta \times \vec{r}_\phi\right| d\phi\, d\theta$$

We need to calculate $\vec{r}_\theta \times \vec{r}_\phi$.

$$\vec{r}_\theta = -2\sin\theta\sin\phi\vec{i} + 2\cos\theta\sin\phi\vec{j}$$
$$\vec{r}_\phi = 2\cos\theta\cos\phi\vec{i} + 2\sin\theta\cos\phi\vec{j} - 2\sin\phi\vec{k}$$

$$\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi \end{matrix}$$

$$= -4\cos\theta\sin^2\phi\vec{i} + 0\vec{j} - 4\sin^2\theta\sin\phi\cos\phi\vec{k} - 4\cos^2\theta\cos\phi\sin\phi\vec{k} - 4\sin^2\phi\cos\theta\vec{j}$$
$$= -4\cos\theta\sin^2\phi\vec{i} - 4\sin^2\phi\cos\theta\vec{j} - 4\sin\phi\cos\phi\vec{k}$$

Thus

$$\left|\vec{r}_\theta \times \vec{r}_\phi\right| = \sqrt{16\cos^2\theta\sin^4\phi + 16\sin^4\phi\cos^2\theta + 16\sin^2\phi\cos^2\phi}$$
$$= \sqrt{16\sin^4\phi + 16\sin^2\phi\cos^2\phi} = 4\sqrt{\sin^2\phi\left(\sin^2\phi + \cos^2\phi\right)}$$
$$= 4\sqrt{\sin^2\phi} = 4\sin\phi$$

So

$$\iint_S z\,dS = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} (2\cos\theta\sin\phi)\left|\vec{r}_\theta \times \vec{r}_\phi\right| d\phi\, d\theta = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} (2\cos\theta\sin\phi)4\sin\phi\, d\phi\, d\theta$$
$$= 8\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} \cos\theta\sin^2\phi\, d\phi\, d\theta$$

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$
$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$
$\int te^t dt = e^t(t-1) + C$
$\int t^2 e^t dt = e^t\left(t^2 - 2t + 2\right) + C$