Ma 227 **Exam III B Solutions**
12/4/12

Name: ________________________________

Lecture Section: ________ **Recitation Section**: _______________

1 [30 **pts**.] Evaluate the surface integral

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS$$

where

$$\vec{F} = y\vec{i} + x\vec{j} + z\vec{k}$$

and S is the surface $3x + 3y + z = 6$ in the first octant oriented with upward normal.
Solution: To parameterize S let $z = 6 - 3x - 3y$ then

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + y\vec{j} + (6 - 3x - 3y)\vec{k}$$

$$\vec{r}_x = 1\vec{i} + 0\vec{j} - 3\vec{k}$$

$$\vec{r}_y = 0\vec{i} + 1\vec{j} - 3\vec{k}$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{vmatrix} = 3\vec{i} + 3\vec{j} + 1\vec{k}$$

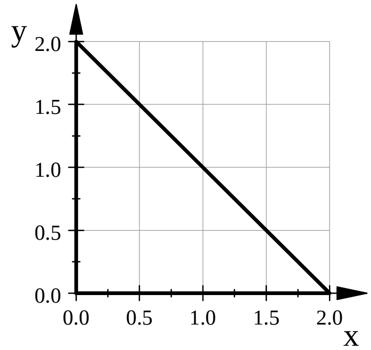
On S

$$\vec{F} = y\vec{i} + x\vec{j} + (6 - 3x - 3y)\vec{k}$$

so

$$\vec{F} \bullet \left(\vec{r}_x \times \vec{r}_y\right) = 3y + 3x + 6 - 3x - 3y = +6$$

The projection of S into the x,y-plane is the line $3x + 3y = 6$ or $y = 2 - x$. The region of integration is the triangle, T, shown below



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Thus

$$\iint_S \vec{F} \bullet \vec{n} dS = \iint_T 6 dA = 6 \cdot (Area\ \ of\ \ T) = 6(\frac{1}{2} \cdot 2 \cdot 2) = 12$$

$$= \int_0^2 \int_0^{2-x} (6) dy dx = 6 \int_0^{\frac{3}{2}} (2-x) dx = 6\left[2x - \frac{x^2}{2} \right]_0^2 = 6(4-2) = 12$$

2 [30 **pts**.] Use Stokes's Theorem to evaluate

$$\int\int_S \text{curl} \vec{F} \cdot \vec{n} ds = \int\int_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n} ds$$

for the vector $\vec{F} = x\vec{i} - y\vec{j}$, where S is the hemisphere $x^2 + y^2 + z^2 = 9$, $z \le 0$. Use an outward normal.

Solution: Stokes Theorem says

$$\int\int_S \text{curl} \vec{F} \cdot \vec{n} ds = \int\int_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n} ds = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

We calculate $\oint_{\partial S} \vec{F} \cdot d\vec{r}$. Now ∂S is the circle $x^2 + y^2 = 9$, $z = 0$. We note that the orientation of the surface corresponds to a clockwise traversal of the boundary circle We parametrize this as

$$x = 3 \sin t, \qquad y = 3 \cot s, \qquad z = 0 \qquad 0 \le t \le 2\pi$$

On ∂S

$$\vec{F} = 3 \sin t \vec{i} - 3 \cos t \vec{j}$$

and

$$\vec{r}(t) = x\vec{i} + y\vec{j} + z\vec{k} = 3 \sin t \vec{i} +.3 \cos t \vec{j} + 0\vec{k}$$
$$\Rightarrow \vec{r}'(t) = 3 \cos t \vec{i} - 3 \sin t \vec{j}$$

Thus,

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (9\sin t \cos t + 9\cos t \sin t)dt = \int_0^{2\pi} 18\sin t \cos t dt = 18\frac{\sin^2 t}{2}\Big|_0^{2\pi} = 0$$

3 $\left[20\ \textbf{pts}.\right]$ Let S be the closed surface of the solid cylinder T bounded by the planes $z = 0$ and $z = 2$ and the cylinder $x^2 + y^2 = 9$.

Calculate the surface integral

$$\iint_S \vec{F} \cdot \vec{n} dS$$

where

$$\vec{F} = \left(x^2 + y^2 + z^2\right)\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

Solution: We will use the Divergence Theorem which is

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_T \nabla \cdot \vec{F} dV.$$

$$\vec{F} = \left(x^2 + y^2 + z^2\right)\left(x\vec{i}\right) + \left(x^2 + y^2 + z^2\right)\left(y\vec{j}\right) + \left(x^2 + y^2 + z^2\right)\left(z\vec{k}\right)$$

$$\text{div}\vec{F} = 3x^2 + y^2 + z^2 + x^2 + 3y^2 + z^2 + x^2 + y^2 + 3z^2 = 5\left(x^2 + y^2 + z^2\right)$$

Then

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_T 5\left(x^2 + y^2 + z^2\right) dV$$

Using cylindrical coordinates to evaluate the integral we have

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_T 5\left(x^2 + y^2 + z^2\right) dV = \int_0^{2\pi}\int_0^3\int_0^2 \left(5\left(r^2 + z^2\right)r\right) dz dr d\theta$$

$$= 5\int_0^{2\pi}\int_0^3 \left[r^3 z + \frac{rz^3}{3}\right]_0^2 dr d\theta = 5\int_0^{2\pi}\int_0^3 \left(2r^3 + \frac{8}{3}r\right) dr d\theta$$

$$= 5\int_0^{2\pi}\left[\frac{2r^4}{4} + \frac{8r^2}{6}\right]_0^3 d\theta = 5\int_0^{2\pi}\left(\frac{81}{2} + 12\right) d\theta$$

$$= 5 \cdot \left(\frac{105}{2}\right) \cdot 2\pi = 525\pi$$

4 [20 **pts**.] If

$$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 1 \end{bmatrix}$$

Find A^{-1}.

Solution: We form $\begin{bmatrix} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{bmatrix} \to_{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{bmatrix} \to_{-2R_1+R_2 \text{ and } -R_1+R_3} \begin{bmatrix} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 & -1 & 1 \end{bmatrix} \to_{R_3+(-1)R_2 \text{ and } R_1+4R_2} \begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix} \to_{-R_2} \begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix} \to_{-R_3} \begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix} \to_{-11R_3+R_1 \;\; 3R_3+R_2} \begin{bmatrix} 1 & 0 & 0 & -7 & 4 & 11 \\ 0 & 1 & 0 & 2 & -1 & -3 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} -7 & 4 & 11 \\ 2 & -1 & -3 \\ 1 & -1 & -1 \end{bmatrix}$

Check (not required) : $AA^{-1} = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} -7 & 4 & 11 \\ 2 & -1 & -3 \\ 1 & -1 & -1 \end{bmatrix} . = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$
$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$
$\int te^t dt = e^t(t-1) + C$
$\int t^2 e^t dt = e^t\left(t^2 - 2t + 2\right) + C$