Ma 227 **Exam III B Solutions** **12/3/13**

Name: ________________________________

1 [30 **pts**.] Evaluate the surface integral

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot \vec{n}\,dS$$

where

$$\vec{F} = 2y\vec{k}$$

and S is the surface with parametrization

$$x = u \quad y = v^2 \quad z = u^2 - v^3 \quad 0 \le u \le 1 \quad 0 \le v \le 1$$

and oriented by an upward pointing normal.

Solution:

$$\vec{r}(u,v) = u\vec{i} + v^2\vec{j} + \left(u^2 - v^3\right)\vec{k}$$

so

$$\vec{r}_u = \vec{i} + 2u\vec{k}$$
$$\vec{r}_v = 2v\vec{j} - 3v^2\vec{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 2v & -3v^2 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & 2u \\ 2v & -3v^2 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 2u \\ 0 & -3v^2 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 0 \\ 0 & 2v \end{vmatrix}$$
$$= 4uv\vec{i} + 3v^2\vec{j} + 2v\vec{k}$$

Since v is non-negative, and the vertical $\left(\vec{k}\right)$ component is $2v\vec{k}$, the vector $\vec{r}_u \times \vec{r}_v$ is an upward normal.

$\vec{F}(u,v) = 2v^2\vec{k}$ so

$$\vec{F}(u,v)\cdot(\vec{r}_u \times \vec{r}_v) = 4v^3$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS = \int_0^1\int_0^1 4v^3\,dv\,du$$
$$= \int_0^1 (1)\,du = 1$$

2 $\left[20\ \textbf{pts.}\right]$ Let S be any closed bounded surface with outward directed normal $\vec{n}$ in x,y,z –space, and

$$\vec{F} = \left(y^2 + xz^2\right)\vec{i} + \left(e^{-x} - yz^2\right)\vec{j} + \cos(x+y)\vec{k}.$$

Evaluate

$$\iint_S \vec{F} \cdot \vec{n} dS$$

Solution: We use the Divergence Theorem

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_T \nabla \cdot \vec{F} dV$$

to evaluate this surface integral.

$$\nabla \cdot \vec{F} = \text{div}\vec{F} = z^2 - z^2 = 0$$

Thus

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_T \nabla \cdot \vec{F} dV = \iiint_T 0 dV = 0$$

where T is the region enclosed by S.

3 [20 **pts**.] Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}.$$

Solution:

$$\det(A - rI) = \begin{vmatrix} 1-r & 2 \\ 2 & -2-r \end{vmatrix} = -(1-r)(2+r) - 4$$

$$= (r-1)(r+2) - 4$$

$$= r^2 + r - 6 = (r+3)(r-2)$$

Thus the eigenvalues are $r = -3, +2$. The system

$$(A - rI)\mathbf{u} = 0$$

is

$$(1-r)u_1 + 2u_2 = 0$$

$$2u_1 + (-2-r)u_2 = 0$$

For $r = -3$ we have

$$4u_1 + 2u_2 = 0$$

$$2u_1 + 1u_2 = 0$$

The elimination process is

$$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $2u_1 = -u_2$. Letting $u_2 = 1$ we have the eigenvector $\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ corresponding to the eigenvalue $r = -3$. Any multiple of this such as $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ may be chosen.

Letting $r = 2$ in the above system we have

$$-u_1 + 2u_2 = 0$$
$$2u_1 - 4u_2 = 0.$$

The elimination step is

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $u_1 = 2u_2$ and an eigenvector corresponding to $r = 2$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

SNB check: $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$, eigenvectors: $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} \leftrightarrow 2, \left\{\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right\} \leftrightarrow -3,$

4 [30 **pts**.] Consider the vector $\vec{F} = -y\vec{i} + 2x\vec{j} + (x+z)\vec{k}$ and let S be the upper hemisphere

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0.$$

with an upward normal.

Note:

$$\operatorname{curl}\vec{F} = 0\vec{i} - \vec{j} + 3\vec{k}$$

Using spherical coordinates since $\rho = 1$, a parametrization of the hemisphere is

$$x = \sin\phi\cos\theta, \qquad y = \sin\phi\sin\theta, \qquad z = \cos\phi$$

With this parametrization

$$\vec{r}(\phi,\theta) = \sin\phi\cos\theta\,\vec{i} + \sin\phi\sin\theta\,\vec{j} + \cos\phi\,\vec{k}$$

and a normal to the hemisphere is

$$\vec{r}_\phi \times \vec{r}_\theta = \sin^2\phi\cos\theta\vec{i} + \sin^2\phi\sin\theta\,\vec{j} + \sin\phi\cos\phi\vec{k}$$

Show this normal is upward, and use this information to verify Stokes' Theorem.
Solution: Stokes Theorem says

$$\int\int_S \operatorname{curl}\vec{F}\cdot\vec{n}ds = \int\int_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}ds = \oint_{\partial S}\vec{F}\cdot d\vec{r}$$

We calculate $\oint_{\partial S}\vec{F}\cdot d\vec{r}$ first. Now ∂S is the circle $x^2 + y^2 = 1$, $z = 0$. We parametrize this as

$$x = \cos t, \qquad y = \sin t, \qquad z = 0 \qquad 0 \leq t \leq 2\pi$$

On ∂S

$$\vec{F} = -\sin t\,\vec{i} + 2\cos t\,\vec{j} + \cos t\vec{k}$$

and

$$\vec{r}(t) = x\vec{i} + y\vec{j} + z\vec{k} = \cos t\,\vec{i} + \sin t\,\vec{j} + 0\vec{k}$$
$$\Rightarrow \vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

Thus,

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (\sin^2 t + 2\cos^2 t)dt = \int_0^{2\pi} (1 + \cos^2 t)dt = \left(t + \frac{1}{2}\cos t \sin t + \frac{1}{2}t\right)_0^{2\pi} = 3\pi$$

We now show that

$$\int\int_S \operatorname{curl}\vec{F} \cdot \vec{n}ds = 3\pi$$

First of all at $\phi = \frac{\pi}{2}$, $\theta = 0$, $\vec{r}_\phi \times \vec{r}_\theta = \vec{i}$, which is upward. Hence $\vec{r}_\phi \times \vec{r}_\theta$ is upward.
Hence

$$\operatorname{curl}\vec{F} \cdot (\vec{r}_\phi \times \vec{r}_\theta) = \left(0\vec{i} - \vec{j} + 3\vec{k}\right) \cdot \left(\sin^2\phi\cos\theta\vec{i} + \sin^2\phi\sin\theta\vec{j} + \sin\phi\cos\phi\vec{k}\right)$$
$$= -\sin^2\phi\sin\theta + 3\sin\phi\cos\phi$$

Thus we have

$$\int\int_S \operatorname{curl}\vec{F} \cdot \vec{n}ds = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} (-\sin^2\phi\sin\theta + 3\sin\phi\cos\phi)d\theta d\phi$$
$$= \int_0^{\frac{\pi}{2}} \left(\sin^2\phi\cos\theta + 3\theta\sin\phi\cos\phi\right)_{\theta=0}^{\theta=2\pi} d\phi$$
$$= 0 + 6\pi\int_0^{\frac{\pi}{2}} \sin\phi\cos\phi d\phi = 6\pi\frac{\sin^2\phi}{2}\Bigg|_0^{\frac{\pi}{2}} = 3\pi \text{ as before.}$$

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$
$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$
$\int te^t dt = e^t(t-1) + C$
$\int t^2 e^t dt = e^t\left(t^2 - 2t + 2\right) + C$