

Name: \_\_\_\_\_

Lecture Section: \_\_\_\_\_ Recitation Section: \_\_\_\_\_

1 [25 pts.] Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}$$

Solution:

$$\begin{aligned} \det(A - rI) &= \begin{vmatrix} 6-r & -3 \\ 2 & 1-r \end{vmatrix} = (6-r)(1-r) + 6 \\ &= r^2 - 7r + 12 = (r-4)(r-3) \end{aligned}$$

Thus the eigenvalues are  $r = 3, 4$ . The system

$$(A - rI)X = 0$$

is

$$(6-r)x_1 - 3x_2 = 0$$

$$2x_1 + (1-r)x_2 = 0$$

For  $r = 3$  we have

$$3x_1 - 3x_2 = 0$$

$$2x_1 - 2x_2 = 0$$

so  $x_1 = x_2$  and an eigenvector is  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . For  $r = 4$  we have

$$2x_1 - 3x_2 = 0$$

$$2x_1 - 3x_2 = 0$$

so  $x_1 = \frac{3}{2}x_2$ . Letting  $x_2 = 2$  yields the eigenvector  $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ . Any multiple of these eigenvectors may also be chosen.

2 [25 pts.] The eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$$

$$\text{are } 1 + 2i \leftrightarrow \begin{bmatrix} 2i \\ 1 \end{bmatrix} \text{ and } 1 - 2i \leftrightarrow \begin{bmatrix} -2i \\ 1 \end{bmatrix}.$$

Give a [real] general solution for the system

$$x' = Ax.$$

Solution: To obtain the required two linearly independent solutions, we start with the complex solution corresponding to one of the pair of complex conjugate eigenvalues and then take the real and imaginary

parts.

$$\begin{aligned} \begin{bmatrix} 2ie^{(1+2i)t} \\ 1e^{(1+2i)t} \end{bmatrix} &= \begin{bmatrix} 2ie^t e^{2it} \\ e^t e^{2it} \end{bmatrix} = \begin{bmatrix} 2ie^t(\cos 2t + i \sin 2t) \\ e^t(\cos 2t + i \sin 2t) \end{bmatrix} \\ &= \begin{bmatrix} -2e^t \sin 2t + 2ie^t \cos 2t \\ e^t \cos 2t + ie^t \sin 2t \end{bmatrix} \end{aligned}$$

Now, we can write the solution in either of the following forms:

$$x = c_1 \begin{bmatrix} -2e^t \sin 2t \\ e^t \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} 2e^t \cos 2t \\ e^t \sin 2t \end{bmatrix} = \begin{bmatrix} -2e^t \sin 2t & 2e^t \cos 2t \\ e^t \cos 2t & e^t \sin 2t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

**3** The eigenvalues and eigenvectors of the matrix

$$B = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$$

$$\text{are } 3 \leftrightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } -1 \leftrightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

**3a [20 pts.]** Find a [particular] solution,  $x_p$ , to

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6t \\ -1 \end{bmatrix}$$

Solution: We seek a solution in the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix}.$$

We substitute this in the d.e. system and solve for the coefficients.

$$\begin{aligned} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} &= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix} + \begin{bmatrix} 6t \\ -1 \end{bmatrix} \\ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= \begin{bmatrix} 6t + ta_1 + 4ta_2 + b_1 + 4b_2 \\ ta_1 + ta_2 + b_1 + b_2 - 1 \end{bmatrix} \\ \begin{bmatrix} (a_1 + 4a_2)t + (-a_1 + b_1 + 4b_2) \\ (a_1 + a_2)t + (-a_2 + b_1 + b_2) \end{bmatrix} &= \begin{bmatrix} -6t \\ 1 \end{bmatrix} \end{aligned}$$

The coefficients of  $t$  give two equations for  $a_1$  and  $a_2$ . We show the row operations.

$$\begin{aligned} \begin{bmatrix} a_1 + 4a_2 \\ a_1 + a_2 \end{bmatrix} &= \begin{bmatrix} -6 \\ 0 \end{bmatrix} \\ \begin{bmatrix} a_1 + 4a_2 \\ -3a_2 \end{bmatrix} &= \begin{bmatrix} -6 \\ 6 \end{bmatrix} \\ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= \begin{bmatrix} 2 \\ -2 \end{bmatrix} \end{aligned}$$

Now, the constants:

$$\begin{aligned} \begin{bmatrix} -a_1 + b_1 + 4b_2 \\ -a_2 + b_1 + b_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} b_1 + 4b_2 \\ b_1 + b_2 \end{bmatrix} &= \begin{bmatrix} a_1 \\ a_2 + 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ \begin{bmatrix} b_1 + 4b_2 \\ -3b_2 \end{bmatrix} &= \begin{bmatrix} 2 \\ -3 \end{bmatrix} \\ \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} &= \begin{bmatrix} -2 \\ 1 \end{bmatrix} \end{aligned}$$

Finally,

$$x_p = \begin{bmatrix} 2t - 2 \\ -2t + 1 \end{bmatrix}$$

**3c [10 pts.]** Give a general solution to

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6t \\ -1 \end{bmatrix}$$

Solution: From the eigenvalues and eigenvectors given at the start of the problem, we can write a fundamental matrix which is

$$\begin{bmatrix} 2e^{3t} & -2e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}.$$

We combine this with the particular solutions from part 3a to provide the result.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2e^{3t} & -2e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 2t - 2 \\ -2t + 1 \end{bmatrix}$$

or

$$\begin{aligned} x_1 &= 2c_1e^{3t} - 2c_2e^{-t} + 2t - 2 \\ x_2 &= c_1e^{3t} + c_2e^{-t} - 2t + 1 \end{aligned}$$

SNB check. 
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6t \\ -1 \end{bmatrix}$$

$$x_1' = x_1 + 4x_2 + 6t$$

$$x_2' = x_1 + x_2 - 1$$

, Exact solution is:  $\left\{ \left[ x_1(t) = 2t + C_3 e^{-t} + C_4 e^{3t} - 2, x_2(t) = \frac{1}{2} C_4 e^{3t} - \frac{1}{2} C_3 e^{-t} - 2t + 1 \right] \right\}$

**4 [20 pts.]** Rewrite the equation

$$y''' + t^2 y'' + e^t y' + 3y = \sin 2t, \quad y(0) = 2, y'(0) = -1, y''(0) = 0$$

as a system of differential equations in normal form with appropriate initial conditions.

Solution: Let

$$x_1(t) = y(t), \quad x_2(t) = y'(t), \quad x_3(t) = y''(t)$$

so

$$x_1'(t) = y'(t) = x_2(t)$$

$$x_2'(t) = y''(t) = x_3(t)$$

$$x_3'(t) = y'''(t) = -t^2 y'' - e^t y' - 3y + \sin 2t = -3x_1 - e^t x_2 - t^2 x_3 + \sin 2t$$

Thus the system is

$$x'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -e^t & t^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin 2t \end{bmatrix}$$

Since

$$x_1(t) = y(t), \quad x_2(t) = y'(t), \quad x_3(t) = y''(t)$$

then the initial condition is

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$