Ma 227 Exam IV B Solutions 12/18/12

Name: ________________________________

Lecture Section: _________ **Recitation Section**: _______________

1 [25 **pts**.] Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 6 & 3 \\ -2 & 1 \end{bmatrix}$$

Solution:

$$\det(A - rI) = \begin{vmatrix} 6-r & 3 \\ -2 & 1-r \end{vmatrix} = (6-r)(1-r)+6$$

$$= r^2 - 7r + 12 = (r-4)(r-3)$$

Thus the eigenvalues are $r = 3, 4$. The system

$$(A - rI)X = 0$$

is

$$(6-r)x_1 + 3x_2 = 0$$
$$-2x_1 + (1-r)x_2 = 0$$

For $r = 3$ we have

$$3x_1 + 3x_2 = 0$$
$$-2x_1 - 2x_2 = 0$$

so $x_1 = -x_2$ and an eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. For $r = 4$ we have

$$2x_1 + 3x_2 = 0$$
$$-2x_1 - 3x_2 = 0$$

so $x_1 = -\frac{3}{2}x_2$. Letting $x_2 = 2$ yields the eigenvector $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Any multiple of these eigenvectors may also be chosen.

SNB check $\begin{bmatrix} 6 & 3 \\ -2 & 1 \end{bmatrix}$, eigenvectors: $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} \leftrightarrow 3, \left\{\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}\right\} \leftrightarrow 4$

2 [25 **pts**.] The eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -9 \\ 1 & 1 \end{bmatrix}$$

are $1 + 3i \leftrightarrow \begin{bmatrix} 3i \\ 1 \end{bmatrix}$ and $1 - 3i \leftrightarrow \begin{bmatrix} -3i \\ 1 \end{bmatrix}$.

Give a [real] general solution for the system

$$x' = Ax.$$

Solution: To obtain the required two linearly independent solutions, we start with the complex solution corresponding to one of the pair of complex conjugate eigenvalues and then take the real and imaginary parts.

$$\begin{bmatrix} 3ie^{(1+3i)t} \\ 1e^{(1+3i)t} \end{bmatrix} = \begin{bmatrix} 3ie^{t}e^{3it} \\ e^{t}e^{3it} \end{bmatrix} = \begin{bmatrix} 3ie^{t}(\cos 3t + i\sin 3t) \\ e^{t}(\cos 3t + i\sin 3t) \end{bmatrix}$$

$$= \begin{bmatrix} -3e^{t}\sin 3t + 3ie^{t}\cos 3t \\ e^{t}\cos 3t + ie^{t}\sin 3t \end{bmatrix}$$

Now, we can write the solution in either of the following forms:

$$x(t) = c_1\begin{bmatrix} -3e^{t}\sin 3t \\ e^{t}\cos 3t \end{bmatrix} + c_2\begin{bmatrix} 3e^{t}\cos 3t \\ e^{t}\sin 3t \end{bmatrix} = \begin{bmatrix} -3e^{t}\sin 3t & 3e^{t}\cos 3t \\ e^{t}\cos 3t & e^{t}\sin 3t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Or we can use the formulas

$$\mathbf{x}_1(t) = e^{\alpha t}(\cos\beta t\mathbf{a} - \sin\beta t\mathbf{b})$$
$$\mathbf{x}_2(t) = e^{\alpha t}(\sin\beta t\mathbf{a} + \cos\beta t\mathbf{b})$$

where $\alpha = 1, \beta = 3$, $\begin{bmatrix} 3i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \mathbf{a} + i\mathbf{b}$. Then

$$\mathbf{x}_1(t) = e^{t}\left(\cos 3t\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin 3t\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right)$$

$$\mathbf{x}_2(t) = e^{t}\left(\sin 3t\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cos 3t\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right)$$

Hence

$$\begin{aligned} x(t) &= +c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) \\ &= \left[c_1 e^{t}\left(\cos 3t\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin 3t\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right) + c_2 e^{t}\left(\sin 3t\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cos 3t\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right)\right] \\ &= c_1\begin{bmatrix} -3e^{t}\sin 3t \\ e^{t}\cos 3t \end{bmatrix} + c_2\begin{bmatrix} 3e^{t}\cos 3t \\ e^{t}\sin 3t \end{bmatrix} \end{aligned}$$

as above.

3 The eigenvalues and eigenvectors of the matrix

$$B = \begin{bmatrix} 1 & 9 \\ 1 & 1 \end{bmatrix}$$

are $4 \leftrightarrow \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $-2 \leftrightarrow \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

3a [20 **pts**.] Find a [particular] solution, x_p, to

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 16t \\ 4 \end{bmatrix}$$

Solution: We seek a solution in the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix}.$$

We substitute this in the d.e. system and solve for the coefficients.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix} + \begin{bmatrix} 16t \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 16t + ta_1 + 9ta_2 + b_1 + 9b_2 \\ ta_1 + ta_2 + b_1 + b_2 + 4 \end{bmatrix}$$

$$\begin{bmatrix} (a_1 + 9a_2)t + (-a_1 + b_1 + 9b_2) \\ (a_1 + a_2)t + (-a_2 + b_1 + b_2) \end{bmatrix} = \begin{bmatrix} -16t \\ -4 \end{bmatrix}$$

The coefficients of t give two equations for a_1 and a_2. We show the row operations.

$$\begin{bmatrix} a_1 + 9a_2 \\ a_1 + a_2 \end{bmatrix} = \begin{bmatrix} -16 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_1 + 9a_2 \\ -8a_2 \end{bmatrix} = \begin{bmatrix} -16 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} a_1 + 9a_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} -16 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -16 - 9(-2) \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Now, the constants:

$$\begin{bmatrix} -a_1 + b_1 + 9b_2 \\ -a_2 + b_1 + b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} b_1 + 9b_2 \\ b_1 + b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 - 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2-4 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} b_1 + 9b_2 \\ -8b_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$$

$$\begin{bmatrix} b_1 + 9b_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2-9(1) \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 1 \end{bmatrix}$$

Fimally,

$$x_p = \begin{bmatrix} 2t-7 \\ -2t+1 \end{bmatrix}$$

3b [10 **pts**.] Give a general solution to

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 16t \\ 4 \end{bmatrix}$$

Solution: From the eigenvalues and eigenvectors given at the start of the problem, namely

$4 \leftrightarrow \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $-2 \leftrightarrow \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

we can write a fundamental matrix which is

$$\begin{bmatrix} 3e^{4t} & -3e^{-2t} \\ e^{4t} & e^{-2t} \end{bmatrix}.$$

We combine this with the particular solutions from part 3a to provide the result.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3e^{4t} & -3e^{-2t} \\ e^{4t} & e^{-2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 2t-7 \\ -2t+1 \end{bmatrix}$$

$$x_1 = 3c_1e^{4t} - 3c_2e^{-2t} + 2t - 7$$
$$x_2 = c_1e^{4t} + c_2e^{-2t} - 2t + 1$$

SNB check.

$$x_1' = x_1 + 9x_2 + 16t$$
$$x_2' = x_1 + x_2 + 4$$

, Exact solution is: $\left\{\left[x_1(t) = 2t + C_1e^{-2t} + C_2e^{4t} - 7, x_2(t) = \frac{1}{3}C_2e^{4t} - \frac{1}{3}C_1e^{-2t} - 2t + 1\right]\right\}$,

4 [20 **pts**.] Rewrite the equation

$$y''' + t^2 y'' + e^t y' + 3y = -\cos 4t,\ y(0) = -2, y'(0) = 1, y''(0) = 0$$

as a system of differential equations in normal form with appropriate initial conditions.

Solution: Let

$$x_1(t) = y(t),\ \ x_2(t) = y'(t),\ x_3(t) = y''(t)$$

so

$$\begin{aligned} x_1'(t) &= y'(t) = x_2(t) \\ x_2'(t) &= y''(t) = x_3(t) \\ x_3'(t) &= y'''(t) = -t^2 y'' - e^t y' - 3y - \cos 4t = -3x_1 - e^t x_2 - t^2 x_3 - \cos 4t \end{aligned}$$

Thus the system is

$$x'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -e^t & t^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\cos 4t \end{bmatrix}$$

Since

$$x_1(t) = y(t),\ \ x_2(t) = y'(t),\ x_3(t) = y''(t)$$

then the initial condition is

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$