Name:________________________________ ID#:________________________

Ma 227 **Final Exam Solutions** **12/12/01**

Name: _______________________________ **ID**: _________________________

Lecture Section: _______________________________ **Lecturer**: _________________________

I pledge my honor that I have abided by the Stevens Honor System.

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Directions: Answer all questions. Each problem is worth **25** points. If you need more work space, continue the problem you are doing on the **other side of the page it is on**. You may *not* use a calculator on this exam.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

#5 ________

#6 ________

#7 ________

#8 ________

Total ________

Name:________________________________ ID#:________________________

Problem 1

a) (**8 points**)
Calculate the iterated integral

$$\int_0^1 \int_{\sqrt{y}}^1 \int_0^y xydzdxdy$$

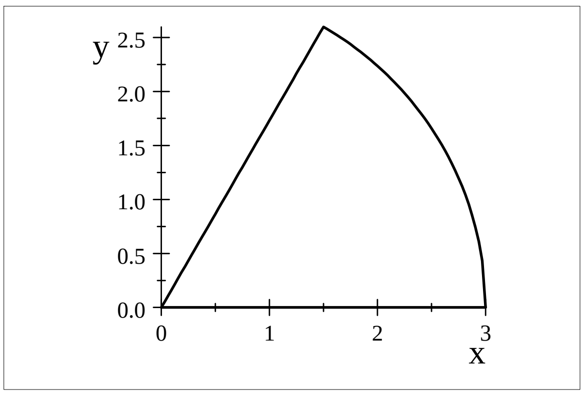
Be sure to show all steps.

$$\int_0^1 \int_{\sqrt{y}}^1 \int_0^y xydzdxdy = \int_0^1 \int_{\sqrt{y}}^1 xy^2dxdy = \int_0^1 \left(\frac{y^2}{2} - \frac{y^3}{2}\right)dy = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$$

b) (**8 points**)
Calculate the value of

$$\int\int_R \left(x^2+y^2\right)^{\frac{3}{2}} dA$$

Where R is the the region in the first quadrant bounded by the lines $y = 0$ and $y = \sqrt{3}\,x$ and the circle $x^2 + y^2 = 9$. Sketch R.
$y = \sqrt{3}\,x$



The line and circle intersect when

$$x^2 + 3x^2 = 9$$

That is at $x = \frac{3}{2}, y = \frac{3\sqrt{3}}{2}$. We use polar coordinates. Then the point of intersection is

$$x = \frac{3}{2} = 3\cos\theta$$

or $\left(3, \frac{\pi}{3}\right)$

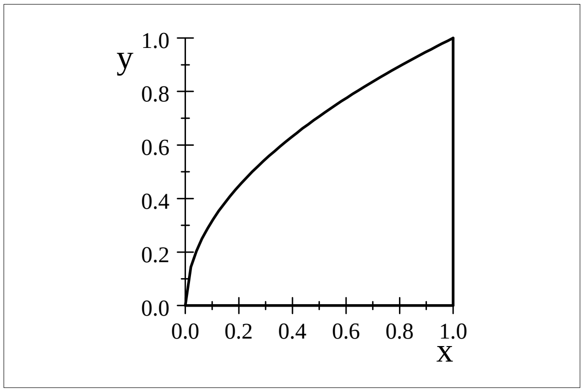
$$\int\int_R \left(x^2+y^2\right)^{\frac{3}{2}} dA = \int_0^{\frac{\pi}{3}} \int_0^3 \left(r^2\right)^{\frac{3}{2}} rdrd\theta = \frac{3^5}{5}\left(\frac{\pi}{3}\right) = \frac{81\pi}{5}$$

c) (**9 points**)
Calculate

$$\int_0^1 \int_{y^2}^1 y\sin\left(x^2\right)dxdy$$

We reverse the order of integration. The region of integration is
$\sqrt{x}$

Name:________________________________ ID#:__________________________



Thus

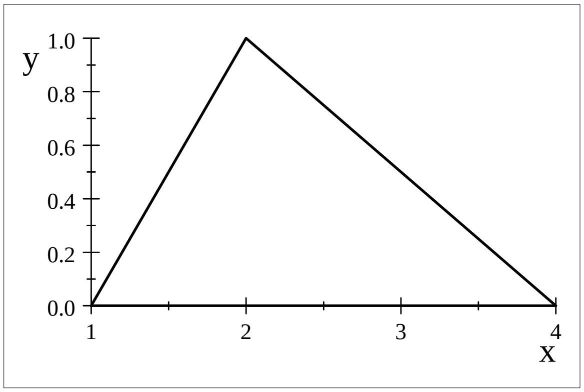
$$\int_0^1\int_{y^2}^1 y\sin\left(x^2\right)dxdy = \int_0^1\int_0^{\sqrt{x}} y\sin\left(x^2\right)dydx = \int_0^1 \frac{x}{2}\sin\left(x^2\right)dx = -\frac{1}{4}\cos\left(x^2\right)\Big|_0^1 = \frac{1}{4}(1-\cos 1)$$

Problem 2

a) (**10 points**)

Give *two* expressions for the volume under the surface $z = x^2y$ and above the triangle in the x, y –plane with vertices $(1,0), (2,1),$ and $(4,0)$. Sketch the triangle, but not the surface. So *not* evaluate the expressions.

$(1,0,2,1,4,0)$



The line joining $(1,0)$ and $(2,1)$ is given by $x - y = 1$ and the line joining $(2,1)$ and $(4,0)$ is given by $x + 2y = 4$. If we call the triangle T, then the volume is given by

$$\text{Volume} = \iint_T \int_0^{x^2y} dzdA$$

$$\iint_T x^2ydA = \int_0^1\int_{1+y}^{4-2y} x^2ydxdy$$

$$= \int_1^2\int_0^{x-1} x^2ydydx + \int_2^4\int_0^{2-\frac{x}{2}} x^2ydydx$$

b) (**15 points**)

Verify Stokes' Theorem is true for the vector field

$$\vec{F}(x,y,z) = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$$

and S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the x,y –plane and S has upward orientation. Sketch S.

We must show

$$\iint_S curl\vec{F} \cdot \vec{n}ds = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = \nabla \times \left(x^2, y^2, z^2\right) = (0,0,0)$$

Thus

$$\iint_S curl\vec{F} \cdot \vec{n}ds = 0$$

For the line integral we parametrize the boundary of $S,$ namely the circle $x^2 + y^2 = 1$ in the x,y –plane, as

$$x = \cos t, \ \ y = \sin t, \quad z = 0 \quad 0 \le t \le 2\pi$$

so

$$\begin{aligned} \vec{r}(t) &= \cos t\vec{i} + \sin t\vec{j} + 0\vec{k} \\ \vec{r}'(t) &= -\sin t\vec{i} + \cos t\vec{j} \\ \vec{F}(t) &= \cos^2 t\vec{i} + \sin^2 t\vec{j} + 0\vec{k} \end{aligned}$$

$$\begin{aligned} \oint_{\partial S} \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \left(-\cos^2 t \sin t + \sin^2 t \cos t\right)dt \\ &= \left[\frac{\cos^3 t}{3} + \frac{\sin^3 t}{3} \right]_0^{2\pi} = 0 \end{aligned}$$

Problem 3

a) (**15 points**)

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution:

Name:________________________________ ID#:________________________

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & -\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (-1)^{3+3}(3-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 1 & -\lambda \end{vmatrix} = -\lambda(3-\lambda)(2-\lambda)$$

Thus the eigenvalues are $\lambda = 0, 2, 3$.
The system $(A - \lambda I)X = 0$ becomes

$$\begin{aligned} (2-\lambda)x_1 &= 0 \\ x_1 - \lambda x_2 + 2x_3 &= 0 \\ (3-\lambda)x_3 &= 0 \end{aligned}$$

For the eigenvalue 0 we have $x_1 = x_3 = 0,$ which gives the eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. For the eigenvalue 3 we have $x_1 = 0, x_2 = \frac{2}{3}x_3$ which yields the eigenvector $\begin{bmatrix} 0 \\ 1 \\ \frac{3}{2} \end{bmatrix}$. For the eigenvalue 2 we have $x_1 - 2x_2 + 2x_3 = 0, x_3 = 0$ which yields the eigenvector $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

b) (**10 points**)
Find a fundamental matrix for the system

$$x'(t) = Ax(t)$$

where A is the matrix above.

$$\begin{bmatrix} 0 & 0 & 2e^{2t} \\ 1 & e^{3t} & e^{2t} \\ 0 & \frac{3}{2}e^{3t} & 0 \end{bmatrix}$$

Problem 4

a) (**13 points**)
Verify Green's theorem when $\vec{F} = 3y\vec{i} - 3x\vec{j}$ and C is the circle $x^2 + y^2 = 16$.
We must show that

$$\oint_C 3y\,dx - 3x\,dy = \iint_{x^2+y^2\le 16} \left(\frac{\partial(-3x)}{\partial x} - \frac{\partial(3y)}{\partial y} \right) dA$$

For the line integral we have

Name:________________________________ ID#:________________________

$$\vec{r}(t) = 4\cos t\vec{i} + 4\sin t\vec{j} \quad 0 \le t \le 2\pi$$
$$\vec{r}'(t) = -4\sin t\vec{i} + 4\cos t\vec{j}$$
$$\vec{F}(t) = 3(4\sin t)\vec{i} - 3(4\cos t)\vec{j}$$

Then

$$\oint_C 3ydx - 3xdy = \int_0^{2\pi} \left(-48\sin^2 t - 48\cos^2 t\right)dt = -48\int_0^{2\pi} dt = -96\pi$$

$$\iint_{x^2+y^2\le 16} \left(\frac{\partial(-3x)}{\partial x} - \frac{\partial(3y)}{\partial y}\right)dA = \iint_{x^2+y^2\le 16} -6dA = -6\left(\pi 4^2\right) = -96\pi$$

b) (12 points)

Does the system of equations

$$x_1 + 2x_2 - 2x_3 + 3x_4 - 4x_5 = 2c$$
$$2x_1 + 4x_2 - 5x_3 + 6x_4 - 5x_5 = 3c$$
$$-x_1 - 2x_2 + 2x_3 - 3x_4 + 4x_5 = 4c$$

have a solution or solutions for all constants c ? If there are solutions, give them.

$$\begin{bmatrix} 1 & 2 & -2 & 3 & -4 & 2c \\ 2 & 4 & -5 & 6 & -5 & 3c \\ -1 & -2 & 2 & -3 & 4 & 4c \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & 3 & -4 & 2c \\ 0 & 0 & -1 & 0 & 3 & -c \\ 0 & 0 & 0 & 0 & 0 & 6c \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & 3 & -4 & 2c \\ 0 & 0 & 1 & 0 & -3 & c \\ 0 & 0 & 0 & 0 & 0 & 6c \end{bmatrix}$$

Thus there is a solution if and only if $c = 0$. In this case the solutions are given by

$$x_3 = 3x_5$$

,

$$x_1 + 2x_2 - 2x_3 + 3x_4 - 4x_5 = 0$$
$$x_1 + 2x_2 + 3x_4 - 10x_5 = 0$$

Problem 5

a) (13 points)

Evaluate $\iint_S \vec{F} \cdot \vec{n}ds,$ where

$$\vec{F}(x,y,z) = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$$

and S is the positively oriented surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and $z = 0$ and $z = 2$ and $\vec{n}$. (Hint: you might want to consider using a theorem.)

Use the Divergence Theorem. Then

$$\iint_S \vec{F} \cdot \vec{n}ds = \iiint_V div\vec{F}dv$$

$$div\vec{F} = 3\left(x^2 + y^2 + z^2\right)$$

Name:__________________________________ ID#:_________________________

$$\iiint_V div\vec{F}dv = \iiint_V 3\left(x^2+y^2+z^2\right)dv = 3\int_0^{2\pi}\int_0^1\int_0^2\left(r^2+z^2\right)rdzdrd\theta$$

$$= 3\int_0^{2\pi}\int_0^1\left(2r^3+\frac{8}{3}r\right)drd\theta = 3\int_0^{2\pi}\left(\frac{1}{2}+\frac{4}{3}\right)d\theta = 3(2\pi)\frac{11}{6} = 11\pi$$

b) (**12 points)**
Consider the

$$\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}}\left(\frac{1}{1+x^2+y^2}\right)dxdy$$

Sketch the region of integration and then give an equivalent expression for this integral in polar coordinates. Do *not* evaluate this expression.

Solution:

x goes from the line $x = y$ to the part of the circle $x^2+y^2 = 4$ in the first quadrant. Thus

The line $x = y$ and the circle $x^2+y^2 = 4$ intersect when $x^2+x^2 = 4,$ that is, at $\left(\sqrt{2},\sqrt{2}\right).$ The angle that the line makes is $\frac{\pi}{4}$

In polar coordinates we have

$$\int_0^{\frac{\pi}{4}}\int_0^2\left(\frac{1}{1+r^2}\right)rdrd\theta$$

Problem 6

a) (**12 points)**
Let $\vec{R} = x\vec{i}+y\vec{j}+z\vec{k}$ and $r = \left|\vec{R}\right|,$ the norm of $\vec{R}.$ Show that

$$\nabla r^n = nr^{n-2}\vec{R} \quad \text{for } n = 1,2,\ldots$$

Solution:

$$r = \left|\vec{R}\right| = \left(x^2+y^2+z^2\right)^{\frac{1}{2}}$$

$$r^n = \left(x^2+y^2+z^2\right)^{\frac{n}{2}}$$

$$\nabla\left(x^2+y^2+z^2\right)^{\frac{n}{2}} = \left(\left(x^2+y^2+z^2\right)^{\frac{1}{2}n} n\frac{x}{x^2+y^2+z^2}, \left(x^2+y^2+z^2\right)^{\frac{1}{2}n} n\frac{y}{x^2+y^2+z^2}, \left(x^2+y^2+z^2\right)^{\frac{1}{2}n} n\frac{}{x^2+}\right.$$

$$= \left(\left(x^2+y^2+z^2\right)^{\frac{1}{2}n-1} nx, \left(x^2+y^2+z^2\right)^{\frac{1}{2}n-1} ny, \left(x^2+y^2+z^2\right)^{\frac{1}{2}n-1} nz\right)$$

$$= nr^{n-2}x\vec{i} + nr^{n-2}y\vec{j} + nr^{n-2}z\vec{k} = nr^{n-2}\vec{R}$$

: :

b) (13 points)

If

$$\vec{F}(x,y,z) = y^2\vec{i} + \left(2xy + e^{3z}\right)\vec{j} + 3ye^{3z}\vec{k}$$

find a function f such that $\nabla f = \vec{F}$.

Solution:

$$f_x = y^2$$

so

$$f = xy^2 + h(y,z)$$

Then

$$f_y = 2xy + h_y = 2xy + e^{3z}$$

Therefore

$$h(y,z) = ye^{3z} + g(z)$$

Then

$$h_z = 3ye^{3z} + g'(z) = 3ye^{3z}$$

This means that $g'(z) = 0,$ so $g = K,$ a constant. Finally

$$f(x,y,z) = xy^2 + ye^{3z} + K$$

Problem 7

a) (8 points)

Find the inverse of

$$P = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 4 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{3}{4} & \frac{1}{4} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}.$$

$$P^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{-1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

b) (5 points)

Show that $D = P^{-1}AP$ where

Name:________________________________ ID#:________________________

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

and P is the matrix in part a) is a diagonal matrix.

$$D = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

c) (**12 points**)

Calculate

$$e^A$$

where A is the matrix in part b).

$$A = PDP^{-1}$$

so

$$e^A = e^{PDP^{-1}} = \sum_{n=0}^{\infty} \frac{\left(PDP^{-1}\right)}{n!} = D + PDP^{-1} + \frac{\left(PDP^{-1}\right)\left(PDP^{-1}\right)}{2!} + \cdots$$

$$= P\left(\sum_{n=0}^{\infty} \frac{D^n}{n!}\right)P^{-1} = Pe^D P^{-1}$$

Since

$$D^n = \begin{bmatrix} 1^n & 0 \\ 0 & 5^n \end{bmatrix}$$

then

$$e^D = \begin{bmatrix} e & 0 \\ 0 & e^5 \end{bmatrix}$$

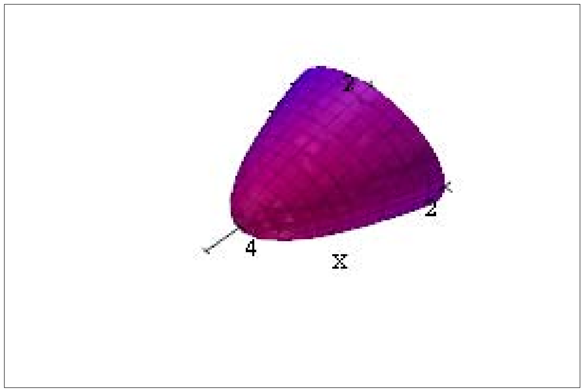
Finally

$$e^A = Pe^D P^{-1} = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e^5 \end{bmatrix}\begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}e + \frac{3}{4}e^5 & -\frac{1}{4}e + \frac{1}{4}e^5 \\ -\frac{3}{4}e + \frac{3}{4}e^5 & \frac{3}{4}e + \frac{1}{4}e^5 \end{bmatrix}$$

Problem 8

a) (**15 points**)

Find the volume of the solid bounded by the plane $x = 0$ and the paraboloid $x = 4 - y^2 - z^2$. Sketch the volume.



$$\text{Volume} = \iint_{0\le y^2+z^2\le 4} \int_0^{4-y^2-z^2} dx dA$$

$$= \iint_{0\le y^2+z^2\le 4} \left(4 - y^2 - z^2\right) dA$$

To evaluate this integral we use polar coordinates in the y, z –plane. Thus $y = r\cos t, z = r\sin t$ so

$$\text{Volume} = \int_0^{2\pi}\int_0^2 \left(4 - r^2\right) r dr d\theta = \int_0^{2\pi}\int_0^2 \left(4r - r^3\right) dr d\theta$$

$$= \int_0^{2\pi} (8 - 4) d\theta = 8\pi$$

b) (**10 points**)

Write the system

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = 6tx_1 - 11x_2 + 6x_3 + \cos t$$

as a single differential equation.

Solution:

Defining

$$x_1 = y, \text{ then } x_2 = y', \text{ and } x_3 = y''$$

Thus

$$y''' = 6ty - 11y' + 6y + \cos t$$