Ma 227 **Final Exam Solutions** **5/9/02**

Name: ________________________________ **ID**: _________________________

Lecture Section: ________________________________

I pledge my honor that I have abided by the Stevens Honor System.

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Directions: Answer all questions. The point value of each problem is indicated. If you need more work space, continue the problem you are doing on the **other side of the page it is on**. You may *not* use a calculator on this exam.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

#5 ________

#6 ________

#7 ________

#8 ________

Total ________

Problem 1

a) (10 points)

Calculate the iterated integral

$$\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{1-x} dzdxdy$$

Be sure to show all steps.

Solution:

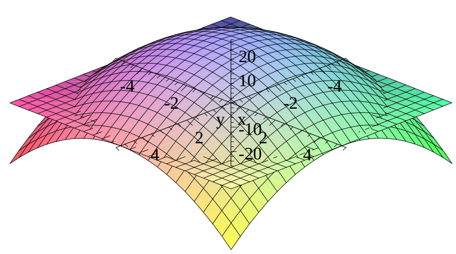
$$\begin{aligned}\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{1-x} dzdxdy &= \int_{-1}^{1}\int_{y^2}^{1}(1-x)dxdy\\ &= -\frac{1}{2}\int_{-1}^{1}(1-x)^2\Big|_{y^2}^{1}dy\\ &= \frac{1}{2}\int_{-1}^{1}\left(1-y^2\right)^2dy\\ &= \frac{1}{2}\int_{-1}^{1}\left(1-2y^2+y^4\right)dy\\ &= \frac{1}{2}\left(y-2\frac{y^3}{3}+\frac{y^5}{5}\right)\Bigg|_{-1}^{1}\\ &= \frac{1}{2}\left(1-\frac{2}{3}+\frac{1}{5}+1-\frac{2}{3}+\frac{1}{5}\right) = \frac{8}{15}\end{aligned}$$

b) (10 points)

Find the volume of the region R bounded below by the the x,y –plane and above by $z = 25 - x^2 - y^2$.
Sketch R.

Solution:

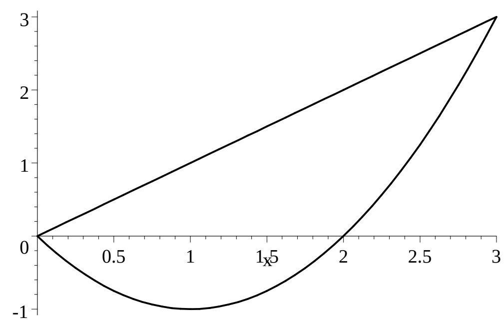
The region is shown in the diagram



The paraboloid intersects the x,y –plane in the circle $x^2+y^2 = 25.$ Thus

$$\begin{aligned}V &= \int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}}\left(25-x^2-y^2\right)dydx\\ &= \int_{0}^{2\pi}\int_{0}^{5}\left(25-r^2\right)rdrd\theta\\ &= \int_{0}^{2\pi}\left(\frac{25r^2}{2}-\frac{r^4}{4}\right)\Bigg|_{0}^{5}d\theta = (2\pi)(625)\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{625\pi}{2}\end{aligned}$$

c) (**10 points**)
Give two integral expressions for the area of the region R bounded by $y = x$ and $y = x^2 - 2x$. Sketch the region R. Do **not** evaluate the integrals.
Solution: The curves intersect when $x^2 - 2x = x$, that is, at $(0,0)$ and $(3,3)$.
The region is shown below.



Thus

$$\int_0^3 \int_{x^2-2x}^{x} dydx$$

or since $y = x^2 - 2x$ implies that $y + 1 = (x-1)^2$ so that $x = 1 \pm \sqrt{y+1}$

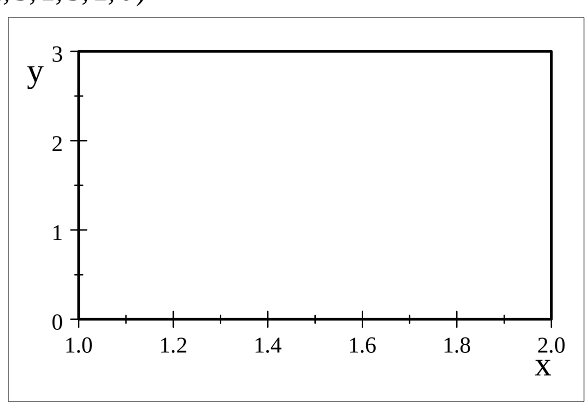
$$\int_{-1}^{0} \int_{1-\sqrt{y+1}}^{1+\sqrt{y+1}} dxdy + \int_0^3 \int_y^{1+\sqrt{y+1}} dxdy$$

Problem 2

a) (**10 points**)
Verify Green's theorem when $\vec{F} = xy\vec{i} - 2xy\vec{j}$ and C is the boundary of the rectangle $1 \le x \le 2$, $0 \le y \le 3$.
Solution: We must show that

$$\oint_C xydx - 2xydy = \iint_{\text{rectangle}} \left(\frac{\partial(-2xy)}{\partial x} - \frac{\partial(xy)}{\partial y} \right) dA$$

The rectangle is $(1,0,2,0,2,3,1,3,1,0)$



Thus

$$\iint\limits_{\text{rectangle}} \left(\frac{\partial(-2xy)}{\partial x} - \frac{\partial(xy)}{\partial y} \right) dA = \int_0^3 \int_1^2 (-2y - x)dxdy = -\frac{27}{2}$$

$$\oint\limits_C xydx - 2xydy = \int_1^2 0dx + \int_0^3 -2(2)ydy + \int_2^1 3xdx + \int_3^0 -2(1)ydy = -\frac{27}{2}$$

b) (**15 points**)

Verify Stokes' Theorem is true for the vector field

$$\vec{F}(x,y,z) = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$$

and S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the x, y –plane and S has upward orientation. Sketch S.

We must show

$$\iint\limits_S curl\vec{F} \cdot \vec{n}ds = \oint\limits_{\partial S} \vec{F} \cdot d\vec{r}$$

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = \nabla \times \left(x^2, y^2, z^2\right) = (0,0,0)$$

Thus

$$\iint\limits_S curl\vec{F} \cdot \vec{n}ds = 0$$

For the line integral we parametrize the boundary of S, namely the circle $x^2 + y^2 = 1$ in the x, y –plane, as

$$x = \cos t, \quad y = \sin t, \quad z = 0 \quad 0 \le t \le 2\pi$$

so

$$\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + 0\vec{k}$$
$$\vec{r}'(t) = -\sin t\vec{i} + \cos t\vec{j}$$
$$\vec{F}(t) = \cos^2 t\vec{i} + \sin^2 t\vec{j} + 0\vec{k}$$

$$\oint\limits_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(-\cos^2 t\sin t + \sin^2 t\cos t\right)dt$$
$$= \left[\frac{\cos^3 t}{3} + \frac{\sin^3 t}{3} \right]_0^{2\pi} = 0$$

Problem 3

a) (**15 points**)

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

Note: $r^3 - 6r^2 + 11r - 6 = (r-3)\left(r^2 - 3r + 2\right)$

Solution: This is example 2 on page 559 of the text.

$$\det \begin{bmatrix} 1-r & 2 & -1 \\ 1 & -r & 1 \\ 4 & -4 & 5-r \end{bmatrix} = -11r + 6r^2 + 6 - r^3 = -(r-1)(r-2)(r-3)$$

Thus the eigenvalues are $r = 1, 2, 3.$ The system of equations for the eigenvectors is

$$\begin{aligned} (1-r)x_1 + 2x_2 - x_3 &= 0 \\ x_1 - rx_2 + x_3 &= 0 \\ 4x_1 - 4x_2 + (5-r)x_3 &= 0 \end{aligned}$$

For $r = 1$ we have

$$\begin{aligned} 2x_2 &= x_3 \\ x_1 - x_2 + x_3 &= 0 \\ 4x_1 - 4x_2 + 4x_3 &= 0 \end{aligned}$$

Thus $x_1 = x_2 - x_3 = -x_2$. Hence the eigenvector corresponding to $r = 1$ is $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. Similarly the other eigenvectors are $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}, \left\{\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}\right\} \leftrightarrow 2, \left\{\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}\right\} \leftrightarrow 3.$

b) (**15 points**)

Solve the initial value problem

$$x'(t) = Ax(t) \qquad x(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

where A is the matrix above.

Solution: This is example 4 on page 563 of the text.

Since the eigenvalues are distinct, then the general solution of the homogeneous DE is

$$x(t) = c_1 e^t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} -c_1 e^t - 2c_2 e^{2t} - c_3 e^{3t} \\ c_1 e^t + c_2 e^{2t} + c_3 e^{3t} \\ 2c_1 e^t + 4c_2 e^{2t} + 4c_3 e^{3t} \end{bmatrix}$$

Thus

$$x(0) = \begin{bmatrix} -c_1 - 2c_2 - c_3 \\ c_1 + c_2 + c_3 \\ 2c_1 + 4c_2 + 4c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} -1 & -2 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ 2 & 4 & 4 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, so $c_1 = 0,\ c_2 = 1, c_3 = -1$ and

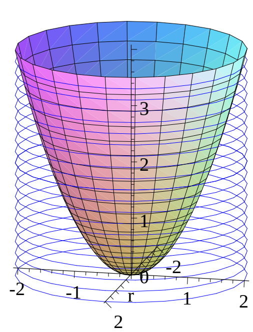
$$x(t) = e^{2t} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} - e^{3t} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

Problem 4

a) (**10 points**)

Give an expression for the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$ and bounded below by the x, y –plane. Sketch the region. Do *not* evaluate the expression.

Solution:



We use cylindrical coordinates. Now z goes from the x, y –plane, that is, 0 to the paraboloid which is $z = r^2$. The region of integration in the x, y –plane is the the circle $x^2 + y^2 \leq 4$, that is $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$. Hence the volume is given by

$$\int_0^{2\pi} \int_0^2 \int_0^{r^2} dz r dr d\theta$$

b) (**12 points**)

Solve, if possible, the system of equations

$$x_1 + 2x_2 - 2x_3 + 3x_4 - 4x_5 = -3$$
$$2x_1 + 4x_2 - 5x_3 + 6x_4 - 5x_5 = -1$$
$$-x_1 - 2x_2 - 3x_4 + 11x_5 = 15$$

Solution: $\begin{bmatrix} 1 & 2 & -2 & 3 & -4 & -3 \\ 2 & 4 & -5 & 6 & -5 & -1 \\ -1 & -2 & 0 & -3 & 11 & 15 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

so a solution does exist and $x_5 = 2, x_3 = 1,$ and $x_1 = 7 - 2x_2 - 3x_4,$ where x_2 and x_4 are arbitrary.

Problem 5

a) (**13 points**)

Evaluate $\iint_S \vec{F} \cdot \vec{n} ds,$ where

$$\vec{F}(x,y,z) = x^3\vec{i} + 2xz^2\vec{j} + 3y^2z\vec{k}$$

and S is the positively oriented surface of the solid bounded by the $z = 4 - x^2 - y^2$ and the x,y –plane, and $\vec{n}$ is the outward directed unit normal to S. (Hint: you might want to consider using a theorem.)

Solution: Use the Divergence Theorem. Then

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_V div\vec{F} dv$$

$$div\vec{F} = 3\left(x^2 + y^2\right)$$

$$\iiint_V div\vec{F} dv = \iiint_V 3\left(x^2 + y^2\right) dV = 3\int_0^{2\pi}\int_0^2\int_0^{4-r^2}\left(r^2\right) rdzdrd\theta$$

$$= 3\int_0^{2\pi}\int_0^2 \left(4 - r^2\right) r^3 drd\theta = 3\int_0^{2\pi}\left(r^4 - \frac{r^6}{6}\right)\Big|_0^2 d\theta = 3(2\pi)\left(16 - \frac{32}{3}\right) = 32\pi$$

b)

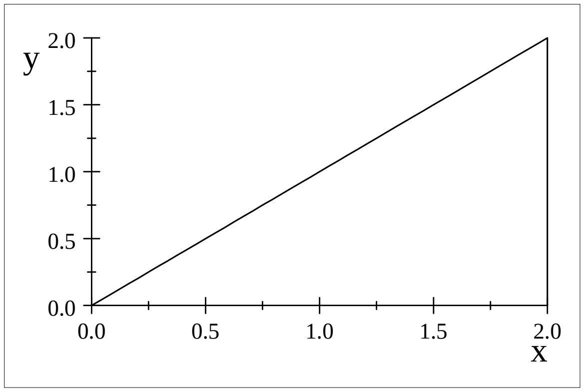
Consider

$$\int_0^2\int_y^2 f(x,y)dxdy.$$

(**5 points**)

(a) Sketch the region of integration.

y



(**5 points**)
(b) Write the integral reversing the order of integration.

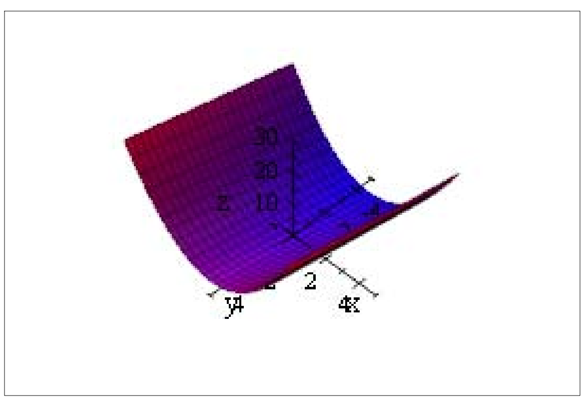
$$\int_0^2 \int_0^x f(x,y)dydx$$

(**7 points**)
(c) Rewrite the integral in terms of polar coordinates.
Solution: The limits on θ are clear from the sketch. Noting that the polar equation of the line $x = 2$ is $r\cos\theta = 2$ or $r = 2\sec\theta$, we have

$$\int_0^{\pi/4} \int_0^{2\sec\theta} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

Problem 6

a) (12 points)
Find the surface area of the part of the surface $z = x + y^2$ that lies above the triangle with vertices $(0,0), (1,1),$ and $(0,1)$.
Solution: The surface projects uniquely onto the region in the x,y –plane as the diagram shows.
$x + y^2$



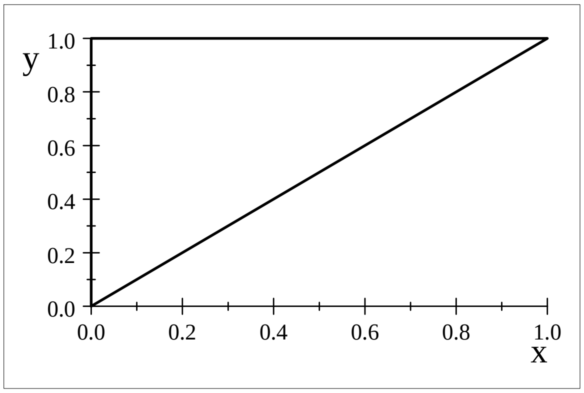
Thus we may use the formula

$$\text{Surface area} = \iint_S \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA$$

where S is given by $z = f(x,y)$. Here $f_x = 1$ and $f_y = 2y$ so

$$\text{Surface area} = \iint_R \sqrt{1+1+4y^2}\, dA$$

where R is the triangle
$(0,0,1,1,0,1,0,0)$



Thus

$$\text{Surface area} = \int_0^1 \int_0^y \sqrt{1+1+4y^2}\, dxdy = \int_0^1 \int_x^1 \sqrt{2+4y^2}\, dydx$$

We use the first expression to find the surface area since this integral can be evaluated.

$$\text{Surface area} = \int_0^1 \int_0^y \sqrt{2+4y^2}\, dxdy = \int_0^1 y\sqrt{2+4y^2}\, dy = \left. \frac{\left(2+4y^2\right)^{\frac{3}{2}}}{\frac{3}{2}(8)} \right|_0^1 = \frac{1}{12}\left(6^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)$$

b) (**13 points**)
If

$$\vec{F}(x,y,z) = \left(2xy^3 + z^2\right)\vec{i} + \left(3x^2y^2 + 2yz\right)\vec{j} + \left(y^2 + 2xz\right)\vec{k}$$

find a function f such that $\nabla f = \vec{F}$.
Solution: Check that such an f exists (not required by problem)

$$\nabla \times \left(2xy^3 + z^2, 3x^2y^2 + 2yz, y^2 + 2xz\right) = (0,0,0)$$

$$f_x = 2xy^3 + z^2$$

so

$$f = x^2y^3 + xz^2 + h(y,z)$$

Then

$$f_y = 3x^2y^2 + h_y = 3x^2y^2 + 2yz$$

Therefore

$$h(y,z) = y^2z + g(z)$$

and

$$f = x^2y^3 + xz^2 + y^2z + g(z)$$

Then

$$f_z = 2xz + y^2 + g'(z) = y^2 + 2xz$$

so $g(z) = K,$ a constant. Thus

$$f = x^2y^3 + xz^2 + y^2z + K$$

Problem 7

a) **(10 points)**

Let A be a constant matrix and r an eigenvalue of A with corresponding eigenvector $u.$ Show that $x(t) = t^r u$ is a solution of the system

$$tx'(t) = Ax(t)$$

Solution: We have that

$$Au = ru$$

Since $x(t) = t^r u,$ then

$$tx'(t) = t\left(rt^{r-1}u\right) = t^r(ru) = t^rAu = A(t^ru) = Ax(t)$$

b) **(15 points)**

Solve the system

$$tx'(t) = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix} x(t) \quad t > 0$$

$\begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}$, eigenvectors: $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\} \leftrightarrow 2, \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \leftrightarrow 4$

Solution: $\det\begin{bmatrix} 1-r & 3 \\ -1 & 5-r \end{bmatrix} = 8 - 6r + r^2 = (r-4)(r-2)$ so the eigenvalues are $2, 4.$

$$\begin{aligned} (1-r)x_1 + 3x_2 &= 0 \\ -x_1 + (5-r)x_2 &= 0 \end{aligned}$$

$r = 2$

$$\begin{aligned} -x_1 + 3x_2 &= 0 \\ -x_1 + 3x_2 &= 0 \end{aligned}$$

so $x_1 = 3x_2$ and the eigenvector is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. The other eigenvector is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \leftrightarrow 4.$ From part a) the solution is

$$x(t) = c_1t^2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2t^4\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Problem 8

(20 points)

Find a **particular** solution of the system

$$x'(t) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -t-1 \\ -4t-2 \end{bmatrix}$$

Solution: This is homework problem #2 on page 579.

$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}, \quad \vec{f}(t) = \begin{bmatrix} -t-1 \\ -4t-2 \end{bmatrix}.$

$\vec{f}(t)$ consists of polynomials of degree 1, our guess will be a vector consisting of polynomials of degree 1: guess $\vec{x}_p(t) = \vec{a}t + \vec{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Now we plug in: $\vec{x}_p'(t) = A\vec{x}_p(t) + \vec{f}(t) \Rightarrow$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) + \begin{bmatrix} -t-1 \\ -4t-2 \end{bmatrix}$$

Bring all of the variables to one side:

$$\begin{bmatrix} t+1 \\ 4t+2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Now equate coefficients:

$t : \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix};$ we now a system of 2 equations in 2 unknowns; solve it any way you like. You should get $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

Now take the constant term on both sides:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, another system of 2 equations in 2 unknowns.

$\Rightarrow \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. So we have $\vec{x}_p(t) = \vec{a}t + \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} t \\ 2 \end{bmatrix}.$