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Ma 227 **Final Exam Solutions** **5/8/03**

Name: ________________________________ **ID**: _________________________

Lecture Section: ________________________________

I pledge my honor that I have abided by the Stevens Honor System.

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Directions: Answer all questions. The point value of each problem is indicated. If you need more work space, continue the problem you are doing on the **other side of the page it is on**. You may *not* use a calculator on this exam.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

#5 ________

#6 ________

#7 ________

#8 ________

Total ________

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Problem 1

a) (10 points)

Calculate the iterated integral

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dzdydx$$

Be sure to show all steps.

Solution:

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dzdxdy = \int_0^1 \int_0^{1-x} (1-x-y)dydx$$

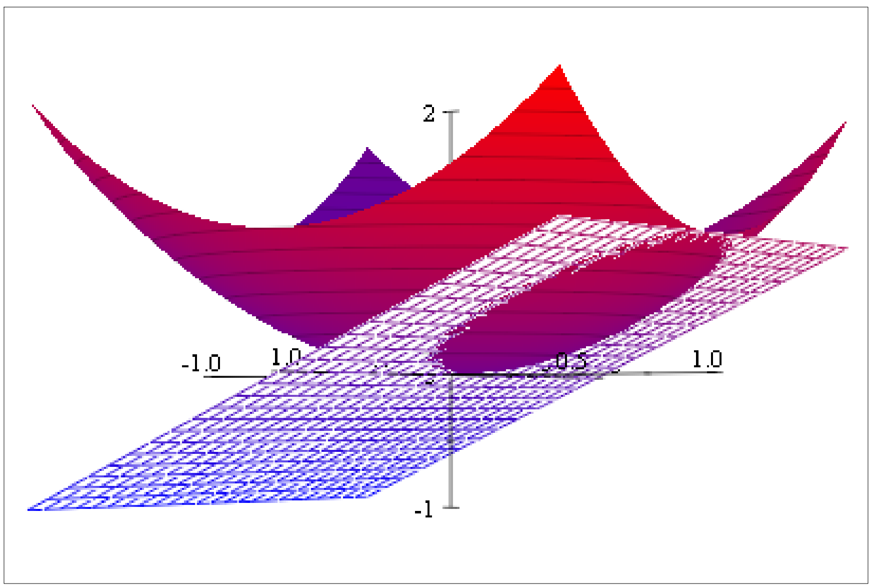
$$= \int_0^1 \left[(1-x)y - \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 (1-x)^2 dx = -\frac{1}{6}\left[(1-x)^3 \right]_0^1 = \frac{1}{6}$$

b) (15 points)

Give an expression in cylindrical coordinates for the volume of the solid T bounded above by the plane $z = y$ and below by the paraboloid $z = x^2 + y^2$. Sketch T. Do *not* evaluate this integral.

y



Solution: In polar coordinates the plane has the equation $z = r\sin\theta$ and the paraboloid has the equation $z = r^2$. The two surfaces intersect when $y = x^2 + y^2$, that is the circle $x^2 + y^2 - y = 0$ or $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$. However, it is only the part of this circle that is in first and second quadrants that is the projection of the solid onto the x,y –plane, since the plane $z = y$ goes through the x axis. The equation of this circle is $r = \sin\theta$

$$\text{Volume} = \int_0^{\pi} \int_0^{\sin\theta} \int_{r^2}^{r\sin\theta} rdzdrd\theta$$

Problem 2

a) (10 points)

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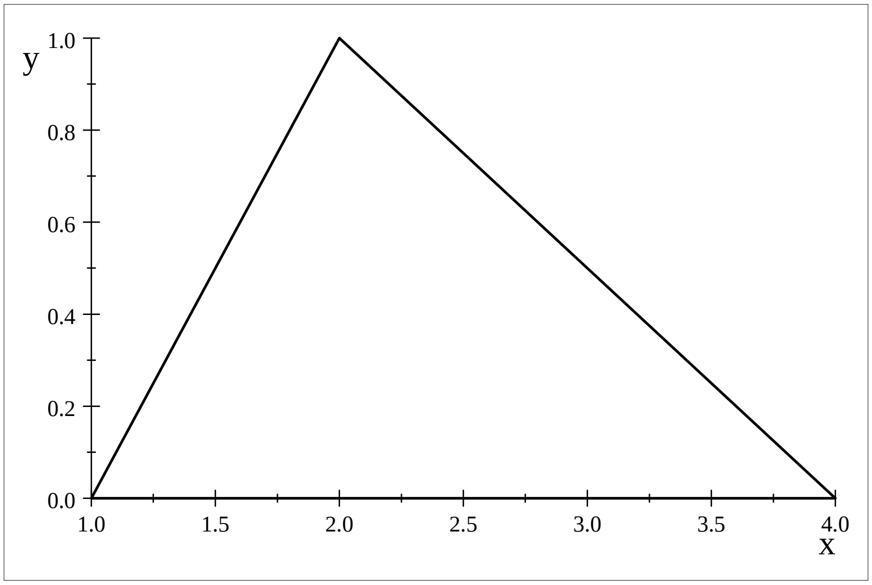
Give **two** triple integral expressions for the volume under the surface $z = x^2 y$ and above the triangle in the x, y –plane with vertices $(1,0),(2,1),(4,0)$. Sketch the triangle in the x, y –plane. Do *not* evaluate the expression.

Solution: $(1,0,2,1,4,0,1,0)$



The line joining $(1,0)$ and $(2,1)$ has equation $y = x - 1$ and the line joining $(2,1)$ to $(4,0)$ has equation $2y = 4 - x$. Thus

$$\text{Volume} = \int_0^1 \int_{y+1}^{4-2y} \int_0^{x^2 y} dz\,dx\,dy$$

$$= \int_1^2 \int_0^{x-1} \int_0^{x^2 y} dz\,dy\,dx + \int_2^4 \int_0^{\frac{4-x}{2}} \int_0^{x^2 y} dz\,dy\,dx$$

b) (10 points)

Calculate the surface integral $\iint_S \vec{F} \cdot \vec{n}\,ds$, where

$$\vec{F}(x,y,z) = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$$

and S is the closed surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2$.

Solution: We can use the Divergence Theorem, since S is a closed surface.

$$\nabla \cdot \vec{F} = 3\left(x^2 + y^2 + z^2\right)$$

so

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$$\iint_S \vec{F}\cdot\vec{n}ds = \iiint_V \nabla\cdot\vec{F}dV$$
$$= \int\int\int 3\left(x^2+y^2+z^2\right)dV$$
$$= 3\int_0^{2\pi}\int_0^1\int_0^2\left(r^2+z^2\right)rdzdrd\theta$$
$$= 3\int_0^{2\pi}\int_0^1\left[r^3z+r\frac{z^3}{3}\right]_0^2 drd\theta$$
$$= 3\int_0^{2\pi}\left[2\frac{r^4}{4}+\frac{8}{3}\left(\frac{r^2}{2}\right)\right]_0^1 d\theta$$
$$= 3\left[\frac{1}{2}+\frac{4}{3}\right](2\pi) = 11\pi$$

Problem 3

a) (**15 points**)

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:

$$\begin{vmatrix} 2-r & 1 & 0 \\ 0 & 3-r & 1 \\ 0 & 0 & 1-r \end{vmatrix} = (2-r)(3-r)(1-r)$$

so the eigenvalues are $r = 1,2,3.$ The system of equations that determines the eigenvectors is

$$(2-r)x_1 + x_2 = 0$$
$$(3-r)x_2 + x_3 = 0$$
$$(1-r)x_3 = 0$$

For $r = 1,$ we have x_3 is arbitrary, and

$$x_1 + x_2 = 0$$
$$2x_2 + x_3 = 0$$

Thus $x_2 = -\frac{1}{2}x_3$ and $x_1 = -x_2 = \frac{1}{2}x_3.$ Thus we have the eigenvector $\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \leftrightarrow 1.$ For $r = 2,$ the system implies, $x_3 = 0, x_2 = 0, x_1$ is arbitrary. Thus $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \leftrightarrow 2.$ For $r = 3$ the system implies

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$x_3 = 0, x_2$ is arbitrary, and $x_1 = x_2$. Thus $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

b) (10 points)

Solve the initial value problem

$$x'(t) = Ax(t) \quad x(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

where A is the matrix above.

Solution: The general solution to the homogeneous system is

$$x(t) = c_1 e^t \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Then

$$x(0) = c_1 \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}c_1 + c_2 + c_3 \\ -\frac{1}{2}c_1 + c_3 \\ c_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Thus $c_1 = 0, c_3 = 0, c_2 = -1$. The solution is

$$x(t) = -e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Problem 4

a) (15 points)

Verify Green's theorem is true for the line integral

$$\oint_C xy^2 dx - x^2 y dy$$

where C consists of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$ and the line segment from $(1, 1)$ to $(-1, 1)$. Sketch C.

Solution:

x^2

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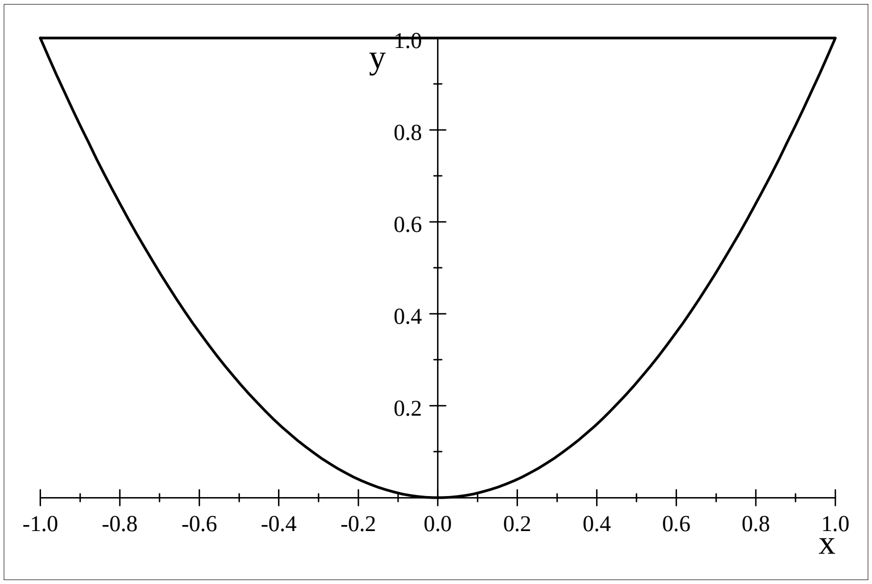

Let C_1 be the parabola and C_2 the line. Then $C_1 : r(t) = t\vec{i} + t^2\vec{j} \quad -1 \le t \le 1$ and $C_2 : r(t) = -t\vec{i} + \vec{j}$ $-1 \le t \le 1$. Thus

$$\oint_C xy^2dx - x^2ydy = \int_{-1}^{1}\left(t \cdot t^4 - t^2 \cdot t^2 \cdot (2t)\right)dt + \int_{-1}^{1} -t(-dt)$$

$$= \left[-\frac{t^6}{6} + \frac{t^2}{2}\right]_{-1}^{1} = 0$$

Using Green's Theorem we have

$$\oint_C xy^2dx - x^2ydy = \iint_R \left[\frac{\partial\left(-x^2y\right)}{\partial x} - \frac{\partial\left(xy^2\right)}{\partial y}\right]dA$$

$$= \iint_R (-2xy - 2xy)dA$$

$$= \int_{-1}^{1}\int_{x^2}^{1}(-4xy)dydx$$

$$= \int_{-1}^{1}\left[-2xy^2\right]_{x^2}^{1}dx = \int_{-1}^{1}(-2x + 2x^5)dx$$

$$= \left[-x^2 + \frac{x^6}{3}\right]_{-1}^{1} = 0$$

b) (10 points)

Put the matrix

$$\begin{bmatrix} 3 & 1 & 9 & -2 \\ 3 & 2 & 12 & 1 \\ 2 & 1 & 7 & -1 \end{bmatrix}$$

in row reduced echelon form.

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Solution: $\begin{bmatrix} 3 & 1 & 9 & -2 \\ 3 & 2 & 12 & 1 \\ 2 & 1 & 7 & -1 \end{bmatrix} \to_{R_1-R_3} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 3 & 2 & 12 & 1 \\ 2 & 1 & 7 & -1 \end{bmatrix} \to_{\substack{-3R_1+R_2 \\ -2R_1+R_3}} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & 6 & 4 \\ 0 & 1 & 3 & 1 \end{bmatrix}$

$$\to_{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 6 & 4 \end{bmatrix} \to_{-2R_2+R_3} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\to_{\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to_{\substack{-R_3+R_2 \\ R_3+R_1}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 5

a) (**10 points**)

Let

$$\vec{F}(x,y,z) = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$$

Calculate

$$\nabla \times \vec{F} = curl\vec{F}$$

Solution: $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = \vec{i}(0-0) + \vec{j}(0-0) + \vec{k}(0-0) = 0$

b) (**15 points**)

Verify that Stokes' Theorem is true for the vector field in part a) where S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the x,y –plane, and S has up orientation.

Solution: Since $\nabla \times \vec{F} = 0$ from part a)

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}dS = 0$$

The bottom of the paraboloid in the x,y –plane is the circle $x^2 + y^2 = 1.$ We let $C : x(t) = \cos t, y(t) = \sin t, z = 0$ so

$$r(t) = \cos t\vec{i} + \sin t\vec{j} + 0\vec{k}, \ \ 0 \le t \le 2\pi$$

$$r'(t) = -\sin t\vec{i} + \cos t\vec{j}$$

$$\vec{F}(t) = \cos^2 t\vec{i} + \sin^2 t\vec{j} + 0\vec{k}$$

Then

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(-\sin t\cos^2 t + \cos t\sin^2 t\right) dt$$

$$= \frac{1}{3}\left[\cos^3 t + \sin^3 t\right]_0^{2\pi} = 0$$

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Problem 6

a) (**15 points**)

If

$$\vec{F}(x,y,z) = \left(20x^3z + 2y^2\right)\vec{i} + 4xy\vec{j} + \left(5x^4 + 3z^2\right)\vec{k}$$

find a function f such that $\nabla f = \vec{F}$.

$$f_y = 4xy$$

so

$$f = 2xy^2 + g(x,z)$$

Then

$$f_z = g_z = 5x^4 + 3z^2$$

so

$$g(x,z) = 5x^4z + z^3 + h(x)$$

Now we have

$$f = 2xy^2 + 5x^4z + z^3 + h(x)$$

$$f_x = 2y^2 + 20x^3z + h'(x) = 20x^3z + 2y^2$$

Hence $h'(x) = 0$ and $h(x) = K.$ Finally,

$$f(x,y,z) = 2xy^2 + 5x^4z + z^3 + K$$

b) (**10 points**)

Evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F}$ is the vector field in part a) and C is the curve given by the vector equation

$$\vec{r}(t) = \left(1 + t^2\right)\vec{i} + (1 + 2t^5)\vec{j} + \left(1 + 3t^6\right)\vec{k} \qquad 0 \le t \le 1$$

Solution: Since there exists a function $f(x,y,z)$ such that $\nabla f = \vec{F},$ the line integral is independent of path. The curve C begins at $(1,1,1)$ and ends at $(2,3,4).$ Therefore

$$\int_C \vec{F} \cdot d\vec{r} = f(2,3,4) - f(1,1,1)$$

$f(2,3,4) = 420 + K$ and $f(1,1,1) = 8 + K.$ Thus

$$\int_C \vec{F} \cdot d\vec{r} = f(2,3,4) - f(1,1,1) = 412$$

Problem 7

a) (**10 points**)

Let A be a constant matrix and r an eigenvalue of A with corresponding eigenvector $u.$ Show that $x(t) = t^r u$ is a solution of the system

$$tx'(t) = Ax(t)$$

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Solution: We have that

$$Au = ru$$

Since $x(t) = t^r u,$ then

$$tx'(t) = t\left(rt^{r-1}u\right) = t^r(ru) = t^r Au = A(t^r u) = Ax(t)$$

b) (**15 points**)
Solve the system

$$tx'(t) = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix} x(t) \quad t > 0$$

Solution: $\det\begin{bmatrix} 1-r & 3 \\ -1 & 5-r \end{bmatrix} = 8 - 6r + r^2 = (r-4)(r-2)$ so the eigenvalues are $2, 4.$

$$(1-r)x_1 + 3x_2 = 0$$
$$-x_1 + (5-r)x_2 = 0$$

$r = 2$

$$-x_1 + 3x_2 = 0$$
$$-x_1 + 3x_2 = 0$$

so $x_1 = 3x_2$ and the eigenvector is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. The other eigenvector is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \leftrightarrow 4.$ From part a) the solution is

$$x(t) = c_1 t^2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 t^4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Problem 8

a) (**15 points**)
Evaluate the

$$\iint_R \sin\theta \, dA$$

where R is the region in the **first** quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta).$ Sketch R and shade it.
Solution:
2

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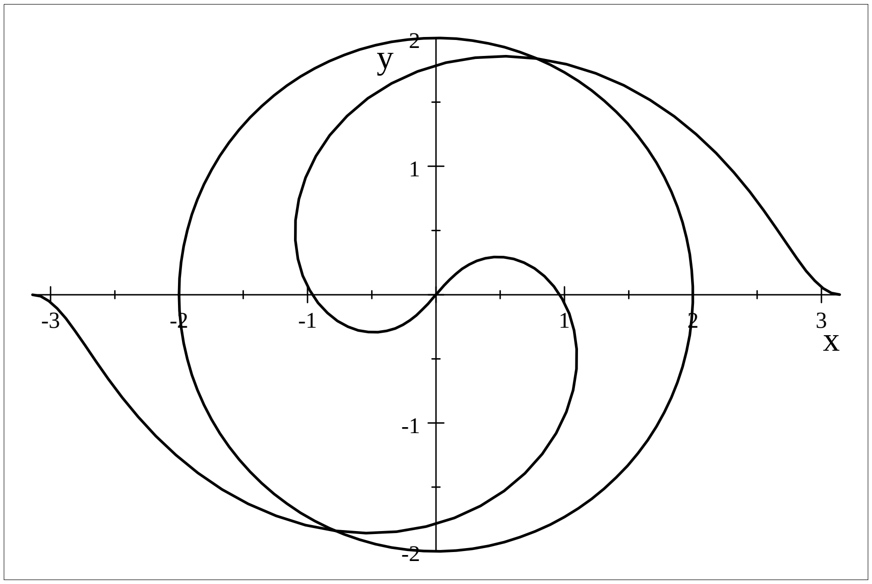

$$\iint_R \sin\theta\, dA = \int_0^{\frac{\pi}{2}} \int_2^{2(1+\cos\theta)} \sin\theta\, r\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{1}{2} r^2 \right]_2^{2(1+\cos\theta)} \sin\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left[(1+\cos\theta)^2 \sin\theta - \sin\theta \right] d\theta = 2\left[-\frac{(1+\cos\theta)^3}{3} + \cos\theta \right]_0^{\frac{\pi}{2}}$$

$$= 2\left[-\frac{1}{3} + \left(\frac{8}{3}\right) - 1 \right] = \frac{8}{3}$$

b) (**10 points**)

Give **two** iterated integrals for the area of the region R in the first quadrant that lies above the hyperbola $xy = 1$ and the line $y = x$ and below the line $y = 2$. Sketch R and shade it. Do **not** evaluate these integrals.

Solution: The hyperbola and the line $y = x$ intersect when $x^2 = 1$, that is at $x = 1$ in the first quadrant. Thus at $(1,1)$. The line $y = 2$ intersects the hyperbola at $x = \frac{1}{2}$, that is at $\left(\frac{1}{2}, 2\right)$. The line $y = x$ intersects the line $y = 2$ at $(2,2)$.

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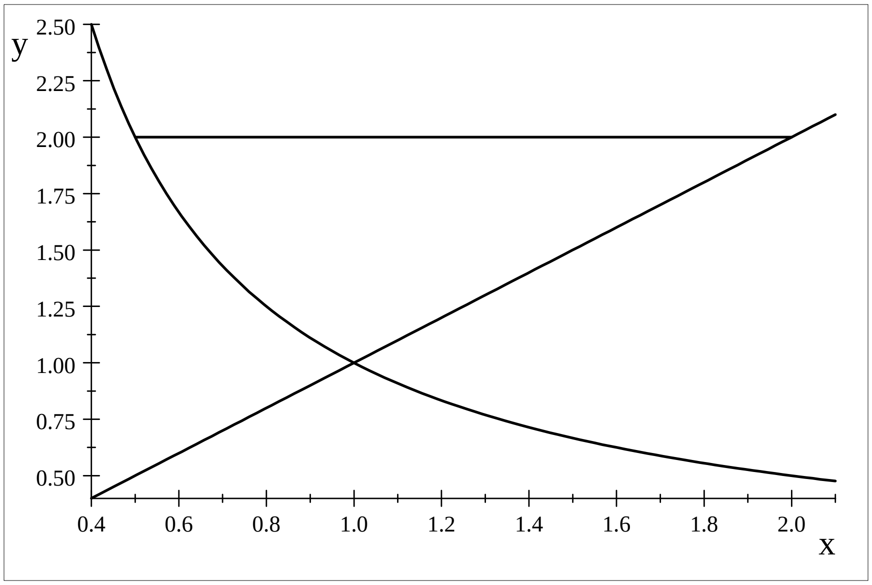

$$\begin{aligned}\text{Area} &= \iint_R dA \\ &= \int_1^2 \int_{\frac{1}{y}}^{y} dxdy \\ &= \int_{\frac{1}{2}}^{1} \int_{\frac{1}{x}}^{2} dydx + \int_1^2 \int_x^2 dydx\end{aligned}$$