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Ma 227 **Final Exam Solutions** **5/13/04**

Name: ________________________________ **ID**: __________________________

Lecture Section: ________________________________

I pledge my honor that I have abided by the Stevens Honor System.

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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Directions: Answer all questions. The point value of each problem is indicated. If you need more work space, continue the problem you are doing on the **other side of the page it is on**.

There is a table of integrals at the end of the exam.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

#5 ________

#6 ________

#7 ________

#8 ________

Total ________

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Problem 1

a) (**13 points**)

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Solution: $\begin{vmatrix} 1-r & 4 \\ 2 & 3-r \end{vmatrix} = (1-r)(3-r) - 8 = 3 - 4r + r^2 - 8 = r^2 - 4r - 5 = (r-5)(r+1)$, so the eigenvalues are $r = -1, 5$.

The system of equations that determines the eigenvectors is $A - rI = 0$ or

$$(1-r)x_1 + 4x_2 = 0$$
$$2x_1 + (3-r)x_2 = 0$$

$r = -1 \Rightarrow$

$$2x_1 + 4x_2 = 0$$
$$2x_1 + 4x_2 = 0$$

so $x_1 = -2x_2$ and an eigenvector corresponding to this eigenvalue is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. $r = 5 \Rightarrow$

$$-4x_1 + 4x_2 = 0$$
$$2x_1 - 2x_2 = 0$$

so $x_1 = x_2$ and an eigenvector corresponding to this eigenvalue is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) (**12 points**)

Solve the nonhomogeneous problem

$$x'(t) = Ax(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where A is the matrix above in 1a).

Solution: A general homogeneous solution is

$$x_h(t) = c_1 e^{-t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We seek a particular solution of the form

$$x_p = \begin{bmatrix} a \\ b \end{bmatrix}$$

Then $x_p' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and the system implies

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$$x_p' = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = Ax_p + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+4b+1 \\ 2a+3b+1 \end{bmatrix}$$

Thus we have the system

$$a + 4b = -1$$
$$2a + 3b = -1$$

, Solution is: $\left\{b = -\frac{1}{5}, a = -\frac{1}{5}\right\}$,, so $x_p = \begin{bmatrix} -\frac{1}{5} \\ -\frac{1}{5} \end{bmatrix}$ and a general solution is

$$x(t) = x_h(t) + x_p(t) = c_1 e^{-t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{5} \\ -\frac{1}{5} \end{bmatrix}$$

SNB check:

$$x_1' = x_1 + 4x_2 + 1$$
$$x_2' = 2x_1 + 3x_2 + 1$$

, Exact solution is: $\begin{bmatrix} x_2(t) = \frac{1}{3}C_1 e^{-t} + \frac{2}{3}C_1 e^{5t} + \frac{1}{3}C_2 e^{5t} - \frac{1}{3}C_2 e^{-t} - \frac{1}{5} \\ x_1(t) = \frac{2}{3}C_1 e^{5t} - \frac{2}{3}C_1 e^{-t} + \frac{2}{3}C_2 e^{-t} + \frac{1}{3}C_2 e^{5t} - \frac{1}{5} \end{bmatrix}$

Problem 2

a) (**12 points**)

Give an integral in cylindrical coordinates for the volume of the solid region that is interior to both the sphere $x^2 + y^2 + z^2 = 4$ of radius 2 and the cylinder $(x-1)^2 + y^2 = 1$. Sketch the region of integration in the x, y –plane. Do *not* evaluate this integral.

Solution: The equation of the cylinder may be simplified to $x^2 + y^2 = 2x$, so in cylindrical coordinates it is $r = 2\cos\theta$. The sphere intersects the x, y –plane as the circle $x^2 + y^2 = 4$ or $r = 2$. Thus the region of integration in the x, y –plane is

2

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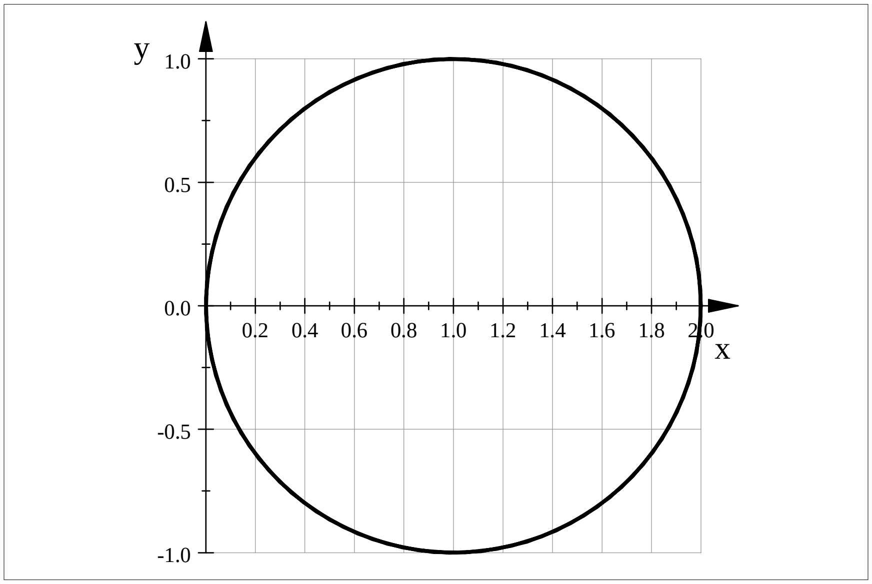


Hence

$$\text{Volume} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz\,r\,dr\,d\theta = \frac{16}{3}\pi - \frac{64}{9}$$

b) (13 points)

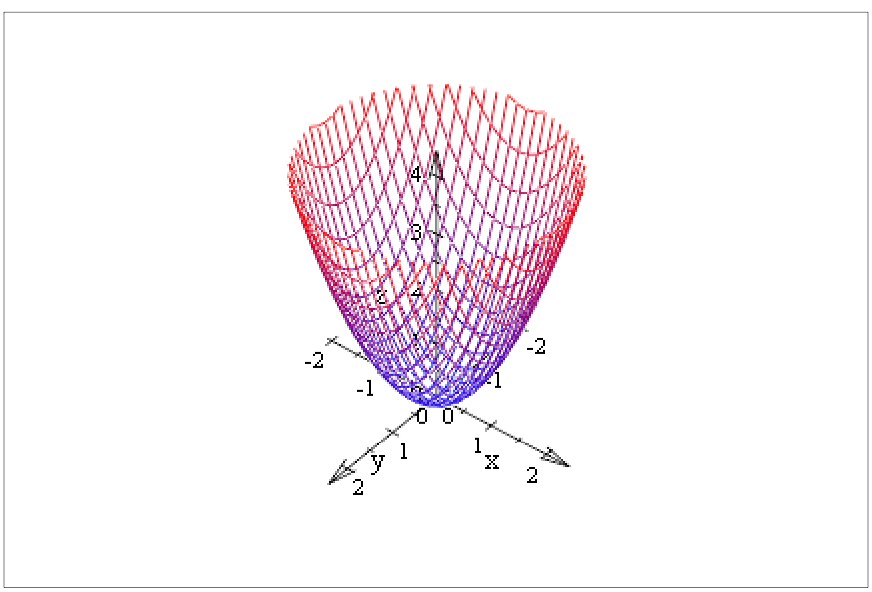
Evaluate the surface integral $\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\,dS,$ where

$$\vec{F}(x,y,z) = 3z\vec{i} + 5x\vec{j} - 2y\vec{k}$$

and S is the part of the parabolic surface $z = x^2 + y^2$ that lies below the plane $z = 4$ and whose orientation is given by the upward unit normal vector.

Solution: The surface is shown below:

$x^2 + y^2$



We use Stokes' Theorem to evaluate this integral where C is the circle $x^2 + y^2 = 4, z = 4,\ 0 \le t \le 2\pi$

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Then

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n} ds = \oint_{x^2+y^2=4} \vec{F} \cdot d\vec{r}$$

C may be parametrized as $x = 2\cos t, y = 2\sin t, z = 4,$ so $\vec{r} = 2\cos t\vec{i} + 2\sin t\vec{j} + 4\vec{k}$ and

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n} ds = \oint_{x^2+y^2=4} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(t) \cdot \vec{r}\,'(t) dt = \int_0^{2\pi} \left(3(4)\vec{i} + 10\cos t\vec{j} - 4\sin t\vec{k}\right) \cdot \left(-2\sin t\vec{i} + 2\cos t\vec{j} + 0\vec{k}\right) dt$$

$$= \int_0^{2\pi} \left(-24\sin t + 20\cos^2 t\right) dt = \int_0^{2\pi} (-24\sin t + 10 + 10\cos 2t) dt = 20\pi$$

Alternatively, we can directly compute the surface integral. First we calculate the integrand.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3z & 5x & -2y \end{vmatrix} = -2\vec{i} + 3\vec{j} + 5\vec{k}$$

For the surface, we use x and y as parameters and have

$$\begin{aligned} \vec{r} &= \left\langle x, y, x^2 + y^2 \right\rangle, 0 \leqq x^2 + y^2 \leqq 4 \\ \vec{r}_x &= \langle 1, 0, 2x \rangle \\ \vec{r}_y &= \langle 0, 1, 2y \rangle \\ \vec{r}_x \times \vec{r}_y &= \langle -2x, -2y, 1 \rangle \end{aligned}$$

Then

$$\begin{aligned} \iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n} dS &= \iint_D \langle -2, 3, 5 \rangle \cdot \langle -2x, -2y, 1 \rangle dA_{xy} \\ &= \iint_D (4x - 6y + 5) dA_{xy} \\ &= \int_0^{2\pi} \int_0^2 (4r\cos\theta - 6r\sin\theta + 5) r dr d\theta \\ &= 20\pi. \end{aligned}$$

Here, the integral is changed into polar coordinates, since the region of integration is the disc $0 \leqq x^2 + y^2 \leqq 4$.

Problem 3

a) (12 points)

Find two iterated integrals representing

$$\iint_R x^2 dA$$

where R is the region bounded by the line $y = x + 2$ and the parabola $y = x^2$. Sketch R. Do *not* evaluate these integrals.

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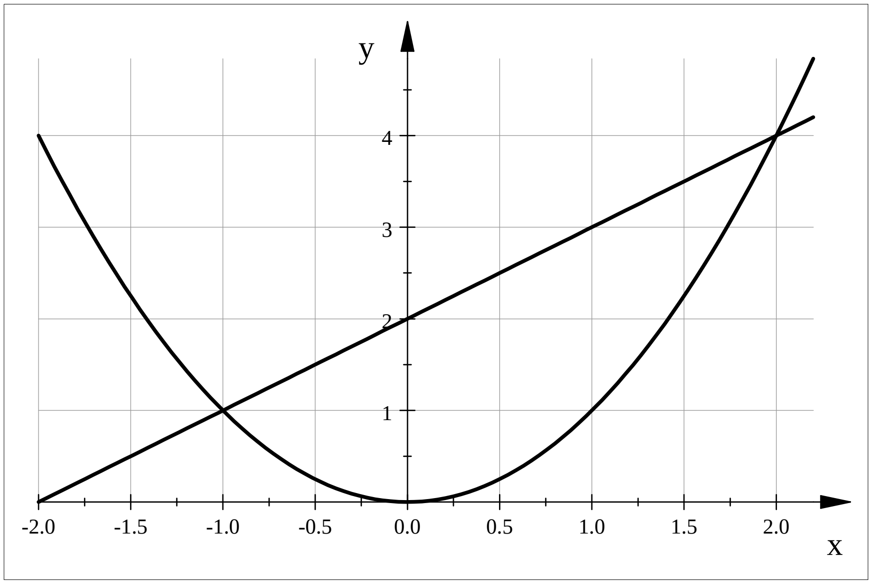
Solution: $x+2$



The two curves intersect when $x^2 = x+2$, so

$$x^2 - x - 2 = (x-2)(x+1) = 0 \Rightarrow x = -1, 2 \text{ and } y = 1, 4 \text{ that is at } (-1,1) \text{ and } (2,4)$$

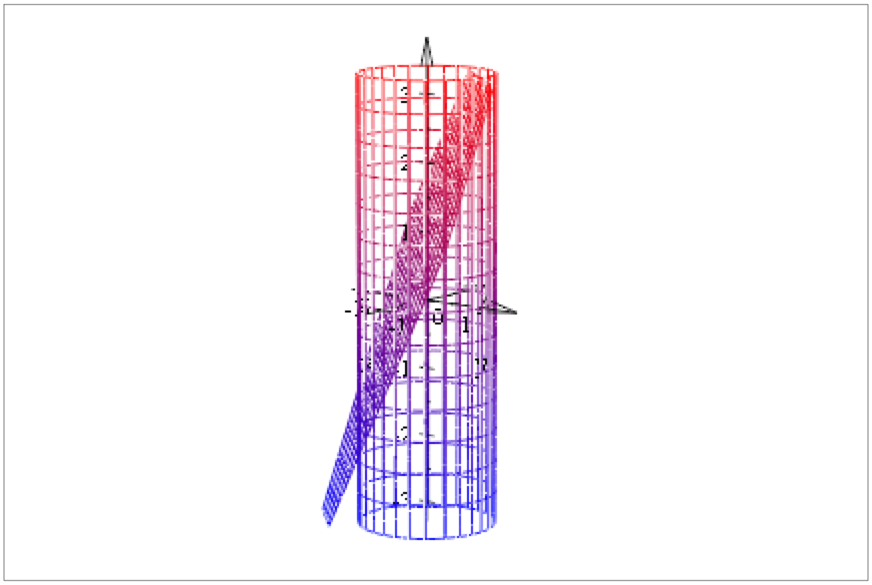
Hence

$$\iint_R x^2 dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} x^2 dx dy + \int_1^4 \int_{y-2}^{\sqrt{y}} x^2 dx dy = \int_{-1}^{2} \int_{x^2}^{x+2} x^2 dy dx = \frac{63}{20}$$

b) (**13 points**)

Find the area of the ellipse cut from the plane $z = 2x + 2y + 1$ by the cylinder $x^2 + y^2 = 1$.

Solution: Below is a picture of the cylinder and the plane.



Since the ellipse projects uniquely onto the x, y –plane, then

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$$\text{Surface Area} = \iint_R \sqrt{1 + (z_x)^2 + (z_y)^2}\, dA$$

where R is the projection of the ellipse onto the x, y –plane, that is the circle $x^2 + y^2 = 1$. Since $z_x = z_y = 2$,

$$\text{Surface Area} = \iint_{x^2+y^2\le 1} \sqrt{1 + (2)^2 + (2)^2}\, dA = \iint_{x^2+y^2\le 1} 3dA = 3 \times \big(\text{area of unit circle}\big) = 3\pi$$

Problem 4

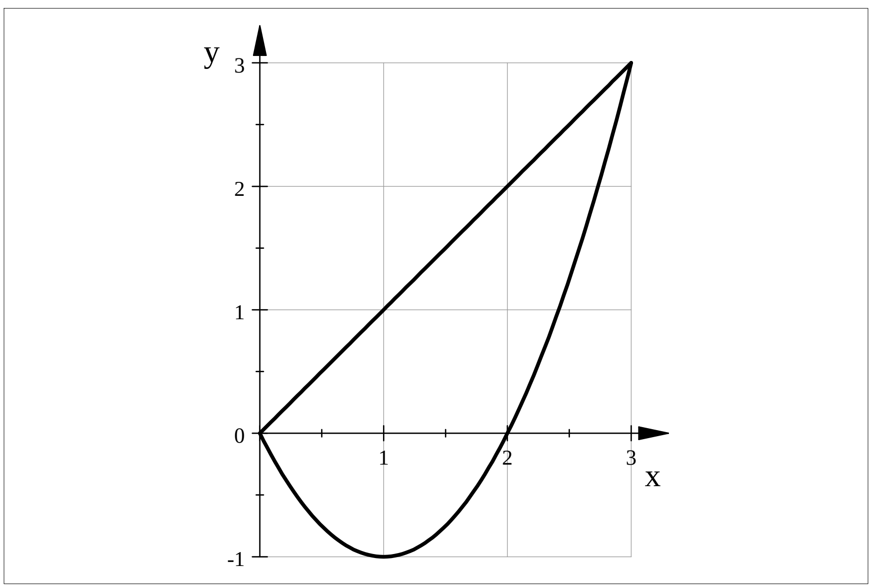
a) (**12 points**)
Evaluate the line integral

$$\oint_C 3xydx + 2x^2dy$$

where C the positively oriented boundary of the region R bounded by the line $y = x$ and the parabola $y = x^2 - 2x$. Sketch C and R.
Solution: We use Green's Theorem

$$\oint_C Pdx + Qdy = \iint_R (Q_x - P_y)dA$$

The region is shown below
$x^2 - 2x$



The curves intersect when $x = x^2 - 2x$, that is when $x^2 - 3x = 0$. Thus the points of intersection are $(0, 0)$ and $(3, 3)$ $Q_x - P_y = 4x - 3x = x$ so

$$\oint_C Pdx + Qdy = \iint_R (Q_x - P_y)dA = \int_0^3 \int_{x^2-2x}^{x} xdydx = \frac{27}{4}$$

Alternatively, we can compute the line integral directly. Since the path is the part of the parabola from the origin to (3,3), followed by the line segment back to the origin, we calculate each part and then add the results.

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On the parabola, C_1,

$$y = x^2 - 2x, \qquad x : 0 \to 3$$
$$dy = (2x-2)dx$$

$$\int_{C_1} 3xydx + 2x^2dy = \int_0^3 \Big(\big(3x\big(x^2-2x\big)\big) + \big(2x^2(2x-2)\big)\Big)dx$$
$$= \int_0^3 \big(7x^3 - 10x^2\big)dx = \frac{207}{4}$$

On the line segment, C_2,

$$y = x, \qquad x : 3 \to 0$$
$$dy = dx$$

$$\int_{C_2} 3xydx + 2x^2dy = \int_3^0 \big(3x^2 + 2x^2\big)dx$$
$$= \int_3^0 5x^2dx = -45$$

Then we add the results to obtain

$$\int_C 3xydx + 2x^2dy = \int_{C_1} 3xydx + 2x^2dy + \int_{C_2} 3xydx + 2x^2dy$$
$$= \frac{207}{4} - \frac{180}{4} = \frac{27}{4}.$$

b) (**13 points**)

Find the inverse of the matrix

$$A = \begin{bmatrix} 4 & 6 \\ 5 & 9 \end{bmatrix}$$

and the use it solve the system $AX = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$.

Solution: $\begin{bmatrix} 4 & 6 & 1 & 0 \\ 5 & 9 & 0 & 1 \end{bmatrix} \to_{-R_1+R_2} \begin{bmatrix} 4 & 6 & 1 & 0 \\ 1 & 3 & -1 & 1 \end{bmatrix} \to_{-4R_2+R_1} \begin{bmatrix} 0 & -6 & 5 & -4 \\ 1 & 3 & -1 & 1 \end{bmatrix}$

$$\to_{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & -6 & 5 & -4 \end{bmatrix} \to_{\left(-\frac{1}{6}\right)R_2} \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -\frac{5}{6} & \frac{2}{3} \end{bmatrix}$$

$$\to_{-3R_2+R_1} \begin{bmatrix} 1 & 0 & \frac{3}{2} & -1 \\ 0 & 1 & -\frac{5}{6} & \frac{2}{3} \end{bmatrix}$$

so $A^{-1} = \begin{bmatrix} \frac{3}{2} & -1 \\ -\frac{5}{6} & \frac{2}{3} \end{bmatrix}$. Since $AX = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$, then

$X = A^{-1}\begin{bmatrix} 6 \\ 18 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -1 \\ -\frac{5}{6} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 6 \\ 18 \end{bmatrix} = \begin{bmatrix} -9 \\ 7 \end{bmatrix}$

Problem 5

a) (13 points)

Let S be the closed surface of the solid cylinder T bounded by the planes $z = 0$ and $z = 3$ and the cylinder $x^2 + y^2 = 4$. Calculate the surface integral

$$\iint_S \vec{F} \cdot ndS$$

where

$$\vec{F} = \left(x^2 + y^2 + z^2\right)\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

Solution: We use the divergence Theorem

$$\iint_S \vec{F} \cdot nds = \iiint_T \nabla \cdot \vec{F} dV$$

$$\vec{F} = \left(x^2 + y^2 + z^2\right)\left(x\vec{i}\right) + \left(x^2 + y^2 + z^2\right)\left(y\vec{j}\right) + \left(x^2 + y^2 + z^2\right)\left(z\vec{k}\right)$$

$$div\vec{F} = 3x^2 + y^2 + z^2 + x^2 + 3y^2 + z^2 + x^2 + y^2 + 3z^2 = 5\left(x^2 + y^2 + z^2\right)$$

Then

$$\iint_S \vec{F} \cdot nds = \iiint_T 5\left(x^2 + y^2 + z^2\right)dV$$

Using cylindrical coordinates to evaluate the integral we have

$$\iint_S \vec{F} \cdot nds = \iiint_T 5\left(x^2 + y^2 + z^2\right)dV = \int_0^{2\pi}\int_0^2\int_0^3 \left(5\left(r^2 + z^2\right)r\right)dzdrd\theta = 300\pi$$

As an alternative, we can calculate the surface integral directly. Since the surface, S, is made up of three components, top (disc), side (cylinder) and bottom (disc), we deal with each components separately and then add the results.

(i) On the top, $z = 3$ and $\vec{n} = \vec{k}$. Thus $\vec{F} = \left(x^2 + y^2 + 9\right)\left(x\vec{i} + y\vec{j} + 3\vec{k}\right)$ and

$$\begin{aligned}\iint_{top} \vec{F} \cdot nds &= \iint_{x^2+y^2 \leqq 4} 3\left(x^2 + y^2 + 9\right)dA \\ &= 3\int_0^{2\pi}\int_0^2 \left(r^2 + 9\right)rdrd\theta \\ &= 3\int_0^{2\pi}\int_0^2 \left(r^3 + 9r\right)drd\theta \\ &= 132\pi.\end{aligned}$$

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(ii) On the bottom, $z = 0$ and $\vec{n} = -\vec{k}$. (Remember, we must use the outward normal.) Thus $\vec{F} = \left(x^2 + y^2\right)\left(x\vec{i} + y\vec{j} + 0\vec{k}\right)$ and

$$\vec{F} \cdot \vec{n} = 0.$$

Hence,

$$\iint_{bottom} \vec{F} \cdot n ds = 0.$$

(iii) On the side, we use cylindrical coordinates to parametrize with $r = 2$. So, we have

$$\begin{aligned}
\vec{r} &= 2\cos\theta\vec{i} + 2\sin\theta\vec{j} + z\vec{k} \\
\vec{r}_\theta &= -2\sin\theta\vec{i} + 2\cos\theta\vec{j} \\
\vec{r}_z &= \vec{k} \\
\vec{r}_\theta \times \vec{r}_z &= -2\sin\theta\vec{i} \times \vec{k} + 2\cos\theta\vec{j} \times \vec{k} \\
&= 2\sin\theta\vec{j} + 2\cos\theta\vec{i}
\end{aligned}$$

Before proceding, we check that we have the correct (outward) normal. This is OK, so we move to the integrand. On the surface, we have

$$\begin{aligned}
\vec{F} &= \left(4 + z^2\right)\left(2\cos\theta\vec{i} + 2\sin\theta\vec{j} + z\vec{k}\right) \\
\vec{F} \cdot (\vec{r}_\theta \times \vec{r}_z) &= \left(4 + z^2\right)\left(4\cos^2\theta + 4\sin^2\theta\right) \\
&= 4\left(4 + z^2\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\iint_{side} \vec{F} \cdot n ds &= \int_0^3 \int_0^{2\pi} 4\left(4 + z^2\right) d\theta dz \\
&= 168\pi
\end{aligned}$$

Finally,

$$\begin{aligned}
\iint_S \vec{F} \cdot n ds &= \iint_{top} \vec{F} \cdot n ds + \iint_{side} \vec{F} \cdot n ds + \iint_{bottom} \vec{F} \cdot n ds \\
&= 132\pi + 168\pi + 0 = 300\pi
\end{aligned}$$

b) (12 points)
Let

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Find e^{Dt}.
Solution: Since D is a diagonal matrix, then

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$$e^{Dt} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix}$$

Problem 6

a) (**13 points**)
Find a function $\phi(x,y,z)$ such that

$$\nabla\phi = \vec{F}(x,y,z) = (2x - y - z)\vec{i} + (2y - x)\vec{j} + (2z - x)\vec{k}$$

Solution:

$$\phi_x = 2x - y - z \quad \phi_y = 2y - x \qquad \phi_z = 2z - x$$

so integrating ϕ_x with respect to x gives

$$\phi = x^2 - xy - xz + g(y,z)$$

Then

$$\phi_y = -x + g_y = 2y - x$$

and $g(y,z) = y^2 + h(z)$. Hence

$$\phi = x^2 - xy - xz + y^2 + h(z)$$

and

$$\phi_z = -x + h'(z) = 2z - x$$

Thus $h(z) = z^2 + K$ and

$$\phi(x,y,z) = x^2 - xy - xz + y^2 + z^2 + K$$

b) (**12 points**)
Evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F}$ is the vector field in part 6a) and C is the curve given by the vector equation

$$\vec{r}(t) = \left(1 + 2t^2\right)\vec{i} + (1 - t^5)\vec{j} + \left(1 + 2t^6\right)\vec{k} \qquad 0 \le t \le 1$$

Solution: Since $\vec{F}$ is derivable from a potential function, then its line integral is path independent. The curve C begins at $(1.1.1)$ and ends at $(3,0,3)$. Thus

$$\int_C \vec{F} \cdot d\vec{r} = \phi(3,0,3) - \phi(1,1,1) = 8$$

Problem 7

a) (**13 points**)
The integral

$$\int_{-2}^{1}\int_{y^2}^{2-y} dxdy$$

gives the area of a region R in the x,y –plane. Sketch R and then give another expression for the area of R with the order of integration reversed. Do *not* evaluate this expression.
Solution: The curves that bound R are the parabola $x = y^2$ and the line $x = 2 - y$ or $y = 2 - x$. Thus

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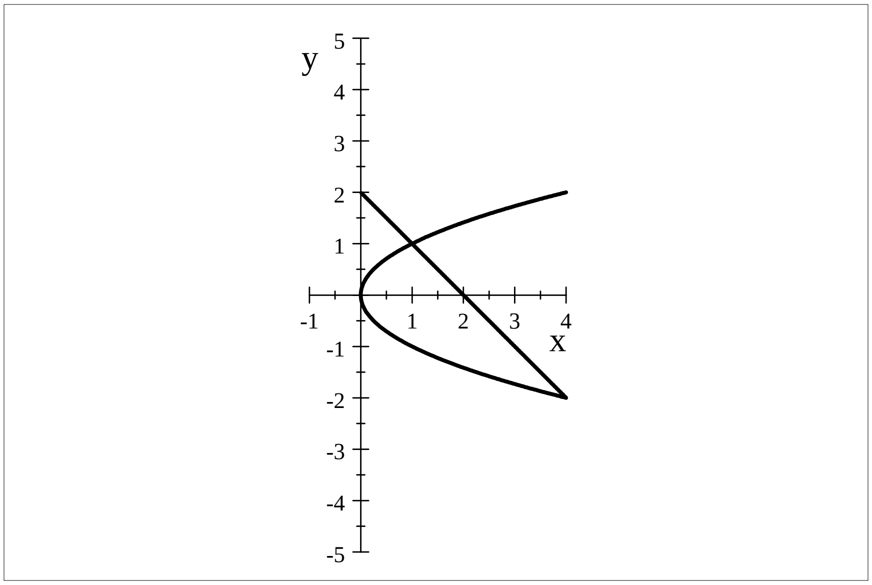
$x = y^2$



The curves intersect when $y^2 = 2 - y$ or when $y^2 + y - 2 = (y+2)(y-1) = 0$. Hence at $(1,1)$ and $(4,-2)$. Reversing the order of integration requires two integrals. They are

$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} dydx + \int_1^4 \int_{-\sqrt{x}}^{2-x} dydx$$

b) (**12 points**)
Evaluate

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\, dydx$$

Sketch the region of integration.

Solution: We use polar coordinates. y goes from 0 to $y = \sqrt{2x - x^2}$. This is part of the circle $x^2 + y^2 = 2x$ or $(x-1)^2 + y^2 = 1$. This is circle with center at $(1,0)$ and radius 1. Since x goes from 0 to 2, we see that this is the upper half of this circle. The equation of the circle in polar coordinates is $r = 2\cos\theta$. Thus the region of integration is

$2\cos\theta$

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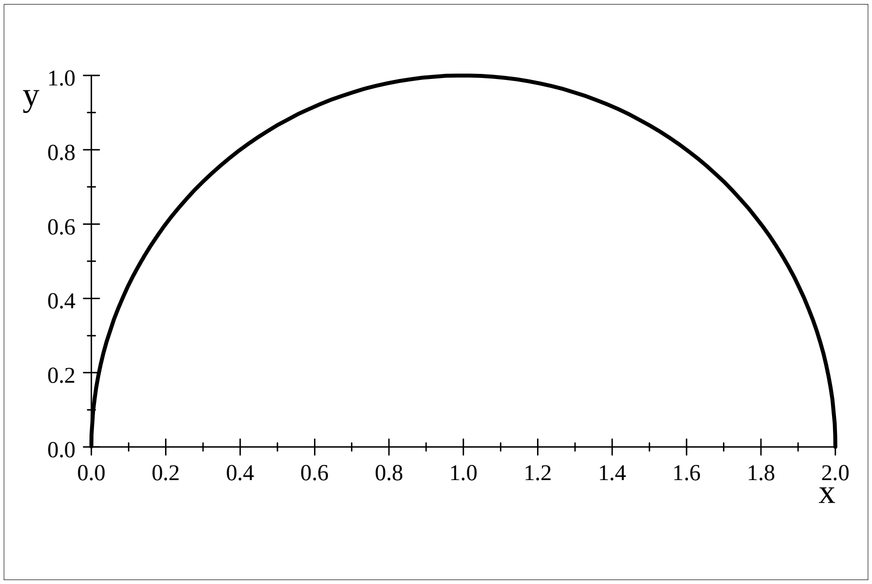

Then

$$\int_0^2\int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\,dydx = \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}(r\sin\theta)r(rdrd\theta) = \int_0^{\frac{\pi}{2}} \sin\theta\left.\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} 16\cos^4\theta\sin\theta d\theta = 4\left[-\frac{1}{5}\cos^5\theta\right]_0^{\frac{\pi}{2}} = \frac{4}{5}$$

The integral is not too difficult to integrate as given.

$$\int_0^2\int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\,dydx = \int_0^2 \frac{1}{3}\left[\left(x^2+y^2\right)^{\frac{3}{2}}\Big|_{y=0}^{y=\sqrt{2x-x^2}}\right. dx$$

$$= \frac{1}{3}\int_0^2\left[\left(x^2+2x-x^2\right)^{\frac{3}{2}} - \left(x^2\right)^{\frac{3}{2}}\right]dx$$

$$= \frac{1}{3}\int_0^2\left[(2x)^{\frac{3}{2}} - x^3\right]dx$$

$$= \frac{1}{3}\left[2^{\frac{3}{2}}\left(\frac{2}{5}x^{\frac{5}{2}}\right) - \frac{1}{4}x^4\right]_{x=0}^{x=2}$$

$$= \frac{4}{5.}$$

Problem 8

a) (**13 points**)

Solve the system

$$x_1 - x_2 - x_3 = 2$$
$$2x_1 - x_2 - 3x_3 = 6$$
$$x_1 - 2x_3 = 4$$

Solution: We form the augmented matrix

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$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -1 & -3 & 6 \\ 1 & 0 & -2 & 4 \end{bmatrix}$$

and row reduce it to reduced echelon form.

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -1 & -3 & 6 \\ 1 & 0 & -2 & 4 \end{bmatrix} \to_{-2R_1+R_2;-R_1+R_3} \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

$$\to_{-R_2+R_3} \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to_{R_2+R_1} \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $x_1 = 2x_3 + 4, x_2 = x_3 + 2$ and x_3 is arbitrary.

b) (**12 points**)

Give one differential equation that is equivalent to the system

$$x'(t) = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 3\cos t \end{bmatrix}$$

Solution: The system is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 3\cos t \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -qx_1(t) - px_2(t) + 3\cos t \end{bmatrix}$$

Thus

$$x_1' = x_2$$
$$x_1'' = x_2' = -qx_1 - px_1' + 3\cos t$$

so the DE is

$$y'' + py' + qy = 3\cos t$$

Name:______________________________ ID#:______________________

Table of Integrals

$$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$$

$$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$$

$$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$$

$$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x$$