Ma 227 **Final Exam Solutions** **12/20/06**

Name: ________________________________

Lecture Section: ________________________________

I pledge my honor that I have abided by the Stevens Honor System.

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You may not use a calculator, **cell phone**, **or computer while taking this exam**. **All work must be shown to obtain full credit**. **Credit will not be given for work not reasonably supported**. **When you finish**, **be sure to sign the pledge**.

Directions: Answer all questions. The point value of each problem is indicated. If you need more work space, continue the problem you are doing on the **other side of the page it is on**.

There is a table of integrals at the end of the exam.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

#5 ________

#6 ________

#7 ________

#8 ________

Total ________

Problem 1

a) (**16 points**)

Compute the curl and divergence of the vector field

$$\vec{F}(x,y,z) = \sin y\vec{i} + x\cos y\vec{j} + (-\sin z + z)\vec{k}$$

Solution:

$$\operatorname{curl}\vec{F} = \left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \sin y & x\cos y & -\sin z + z & \sin y & x\cos y \end{array}\right|$$

$$= 0\vec{i} + 0\vec{j} + \cos y\vec{k} - \cos y\vec{k} - 0\vec{i} - 0\vec{j} = 0$$

$$\operatorname{div}\vec{F} = -x\sin y - \cos z + 1$$

SNB check $\nabla \times (\sin y, x\cos y, -\sin z + z) = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$ $, \nabla \bullet (\sin y, x\cos y, -\sin z + z) : 1 - x\sin y - \cos z$

b) (**9 points**)

Does there exist a function $\phi(x,y,z)$ such that $\nabla\phi = \vec{F}$ where $\vec{F}$ is the vector field in 1a above. Why or why not? If yes. then find $\phi(x,y,z)$

Solution: Since $\operatorname{curl}\vec{F} = 0,$ such a $\phi(x,y,z)$ exits.

$$\phi_x = \sin y \quad \phi_y = x\cos y \quad \phi_z = -\sin z + z$$

Integrating ϕ_x yields

$$\phi(x,y,z) = x\sin y + g(y,z)$$

so

$$\phi_y = x\cos y + \frac{\partial g}{\partial y} = x\cos y$$

Hence $g(y,z) = h(z)$ and

$$\phi(x,y,z) = x\sin y + h(z)$$

Therefore

$$\phi_z = h'(z) = -\sin z + z$$

and

$$h(z) = \cos z + \frac{z^2}{2} + K$$

and we have

$$\phi(x,y,z) = x\sin y + \cos z + \frac{z^2}{2} + K$$

SNB Check: $\nabla\left(x\sin y + \cos z + \frac{z^2}{2} + K\right) = \begin{matrix} \sin y \\ x\cos y \\ z - \sin z \end{matrix}$

Problem 2

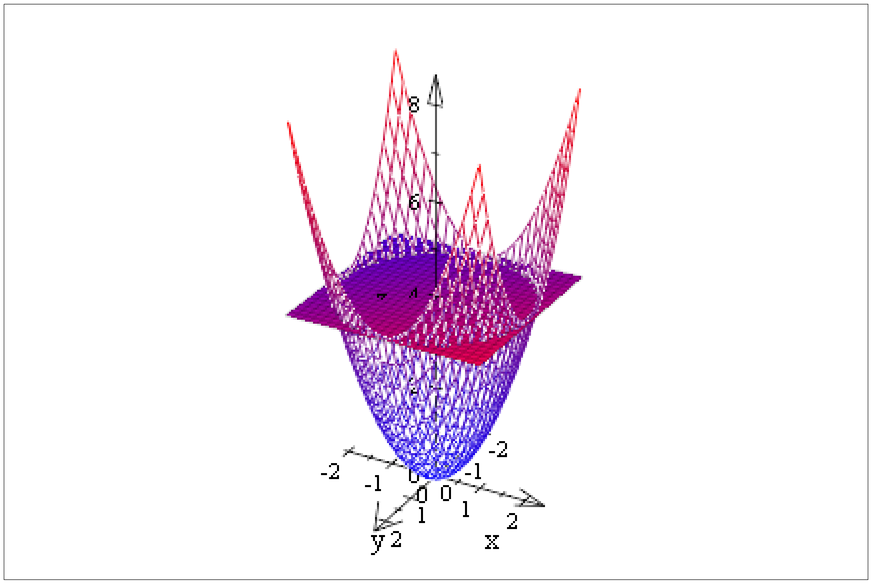
a) (**12 points**)

Give an expression in polar coordinates for

$$\iint_S z\,ds$$

where S is the part of the paraboloid $z = x^2 + y^2$ under the plane $z = 4$. Sketch S. Do *not* evaluate the expression.

Solution: $x^2 + y^2$



Since the surface can be uniquely projected onto the disc $x^2 + y^2 \leq 4$ in the x, y –plane, we may use the expression

$$\iint_S f(x,y,z)ds = \iint_R f(x,y,z(x,y))\sqrt{1 + (z_x)^2 + (z_y)^2}\,dA$$

$$\iint_S z\,ds = \iint_{x^2+y^2\leq 4} \left(x^2 + y^2\right)\sqrt{1 + (2x)^2 + (2y)^2}\,dA$$

$$= \int_0^{2\pi}\int_0^2 r^2\sqrt{1 + 4r^2}\,r\,dr\,d\theta$$

Alternate solution:

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + \left(x^2 + y^2\right)\vec{k}$$

$$\vec{r}_x = \vec{i} + 2x\vec{k}$$

$$\vec{r}_y = \vec{j} + 2y\vec{k}$$

$$\vec{r}_x \times \vec{r}_y = (1,0,2x) \times (0,1,2y) = \begin{pmatrix} -2x & -2y & 1 \end{pmatrix} = -2x\vec{i} - 2y\vec{j} + \vec{k}$$

Thus

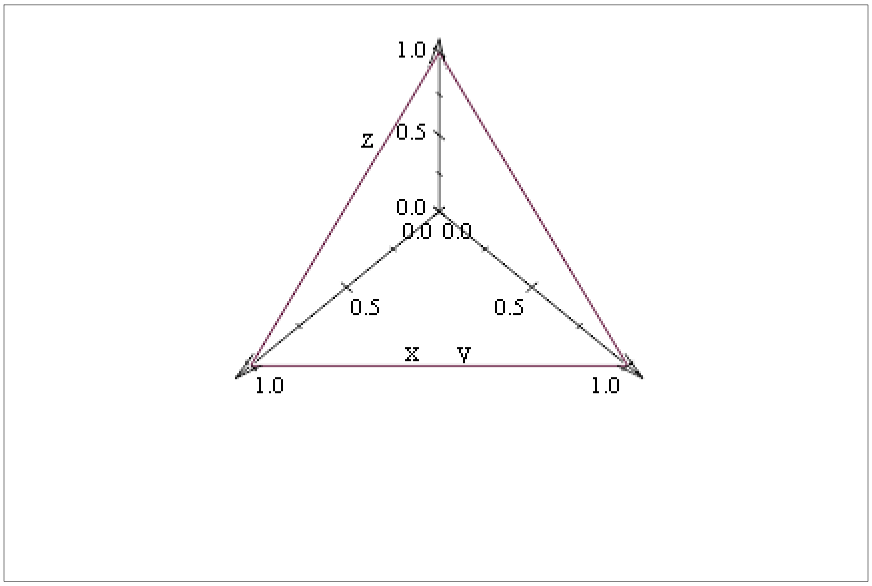
$$\iint_S z\,ds = \iint_S z|\vec{r}_x \times \vec{r}_y| = \iint_{x^2+y^2\le 4} \left(x^2+y^2\right)\sqrt{1+(2x)^2+(2y)^2}\,dA$$

as before.

b) (**13 points**)

Use Stokes' Theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ and C is the triangle with vertices $(1,0,0), (0,1,0),$ and $(0,0,1),$ oriented counter-clockwise when viewed from above.

Solution: $(1,0,0,0,1,0,0,0,1,1,0,0)$



The surface is given by

$$x + y + z = 1$$

Thus a parametrization is given by

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + (1-x-y)\vec{k}$$

$$\vec{r}_x = \vec{i} - \vec{k} \qquad r_y = \vec{j} - \vec{k}$$

so

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 1 & 0 \\ 0 & 1 \end{matrix} = \vec{k} + \vec{i} + \vec{j}$$

This is clearly outer. Also since $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xy & yz \end{matrix} = -x\vec{k} - y\vec{i} - z\vec{i}$$

Therefore

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F} \cdot dS = \iint_S \operatorname{curl}\vec{F} \cdot \vec{n}ds$$

$$= \iint_S \left(-x\vec{k} - y\vec{i} - z\vec{i}\right) \cdot \left(\vec{k} + \vec{i} + \vec{j}\right)ds = \iint_R (-x - y - z)dA$$

$$= \iint_R (-1)dA = -(\text{area of } R)$$

since on the $x + y + z = 1$. Now R is the triangle $x + y = 1$ in the x, y –plane and it has area $\frac{1}{2}$. Thus

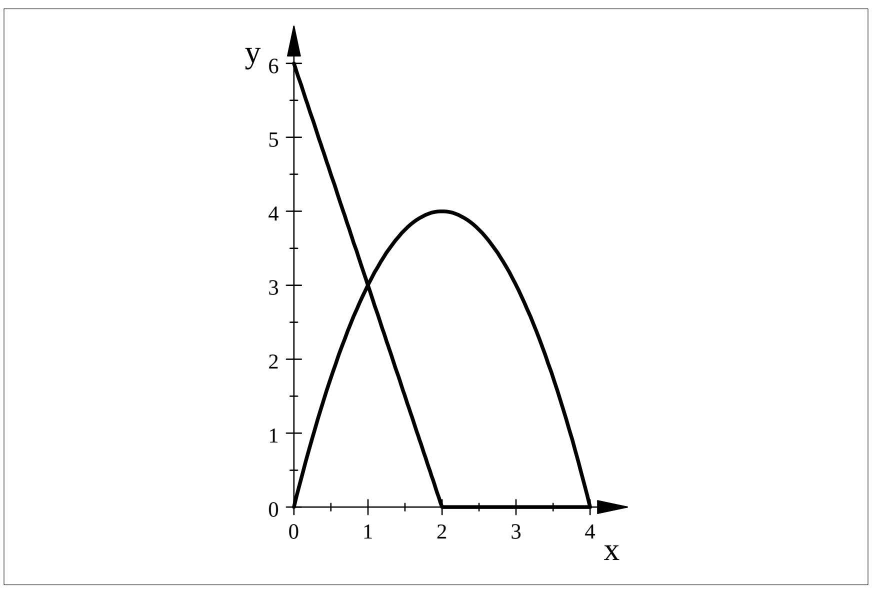
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F} \cdot dS = \iint_S \operatorname{curl}\vec{F} \cdot \vec{n}ds = -\frac{1}{2}$$

Problem 3

a) (12 points)

Let R be the region that lies below the parabola $y = 4x - x^2$ above the x –axis and above the line $y = -3x + 6$. Sketch R. Give two integrals with different orders of integration for the area of R. Do *not* evaluate either of these integrals.

Solution: $4x - x^2$



Solution: The parabola can be rewritten as $y = -(x-2)^2 + 4$. solving for x we have $(x-2)^2 = 4 - y$ or $x = \pm\sqrt{4-y} + 2$. For line we have $x = \frac{y-6}{-3}$.

The line and the parabola intersect when

$$-3x + 6 = 4x - x^2$$

or

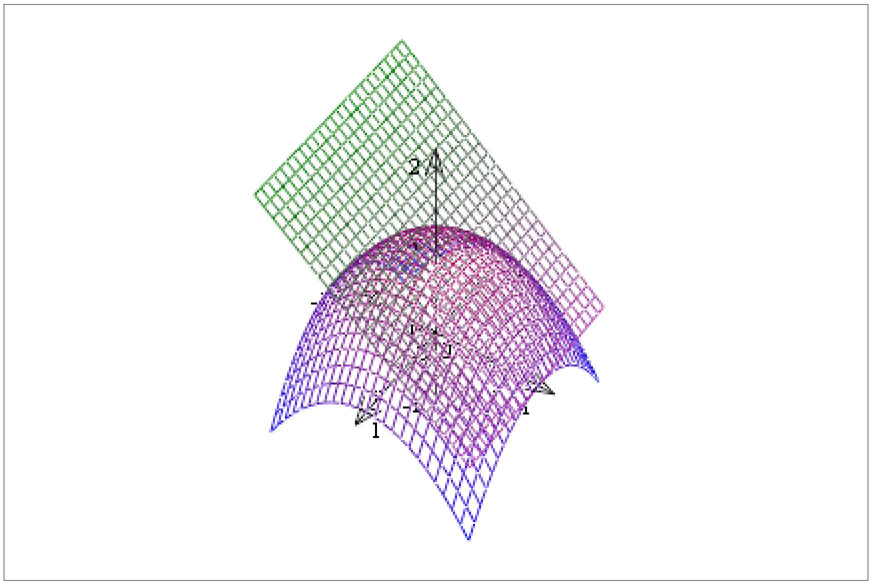
$$x^2 - 7x + 6 = (x-6)(x-1) = 0$$

$x = 1$ is in the region so they intersect at $(1,3)$. Thus

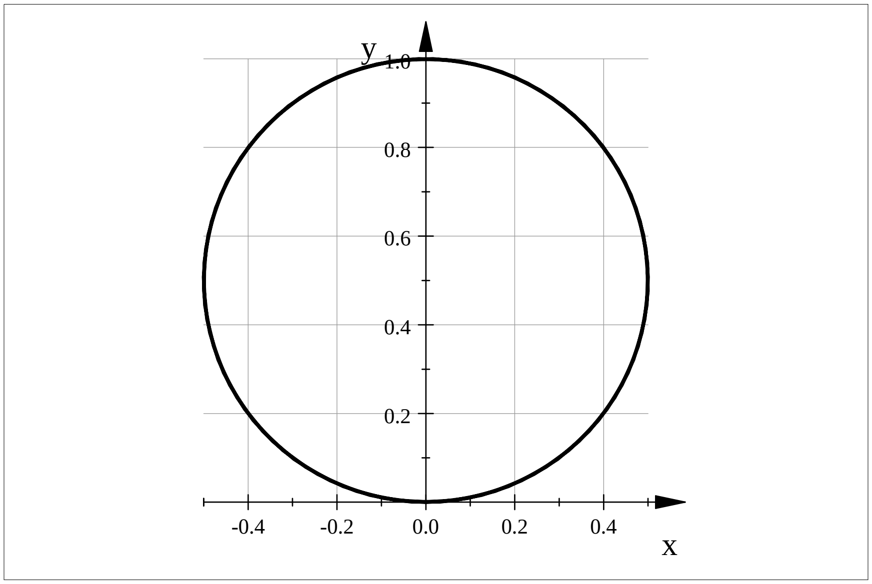
$$\text{Area of } R = \int_1^2 \int_{-3x+6}^{4x-x^2} dydx + \int_2^4 \int_0^{4x-x^2} dydx$$
$$= \int_0^3 \int_{\frac{y-6}{-3}}^{+\sqrt{4-y}+2} dxdy + \int_3^4 \int_{-\sqrt{4-y}+2}^{+\sqrt{4-y}+2} dxdy$$

b) (**13 points**)

Give an expression in cylindrical coordinates for the volume of the solid region D bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the plane $z = 1 - y$. Do *not* evaluate this expression.
Solution: $1 - x^2 - y^2$



The two surfaces intersect when $1 - y = 1 - x^2 - y^2$ or $x^2 + y^2 - y = 0$. This is a circle in the x,y –plane, namely $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$. In polar coordinates the equation of the circle is $r = \sin\theta$. The region of integration in the x,y –plane is shown below.
$\sin\theta$



z goes from the plane to the paraboloid. Thus the volume is given by

$$\int_0^{\pi}\int_0^{\sin\theta}\int_{1-r\sin\theta}^{1-r^2} r\,dz\,dr\,d\theta$$

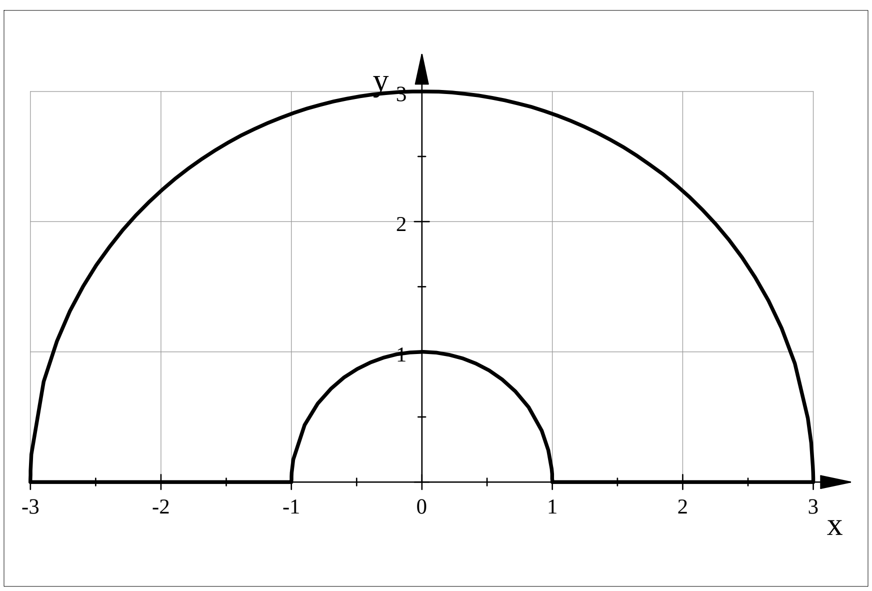
Problem 4

a) (15 points)

Evaluate the line integral

$$\oint_C \left(\arctan x + y^2\right)dx + \left(e^y - x^2\right)dy$$

where C is the path enclosing the annular region shown below.



Solution: We use Green's Theorem. Here $P = \arctan x + y^2$ and $Q = e^y - x^2$. The $Q_x - P_y = -2x - 2y$ so

$$\oint_C \left(\arctan x + y^2\right)dx + \left(e^y - x^2\right)dy = \iint_R -2(x+y)\,dA$$

$$= -2\int_0^{\pi}\int_1^{3} r(\cos\theta + \sin\theta)\,r\,dr\,d\theta$$

$$= -2\int_0^{\pi}(\cos\theta + \sin\theta)\left.\frac{r^3}{3}\right]_1^3$$

$$= -\frac{2}{3}(26)[\sin\theta - \cos\theta]_0^{\pi} = -\frac{104}{3}$$

b) **(10 points)**

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}.$$

Solution: $$\begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2+R_1} \begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 4 & 1 & 1 & 0 \\ 0 & -8 & -7 & -3 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{4}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -8 & -7 & -3 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[8R_2+R_3]{-3R_2+R_1} \begin{bmatrix} 1 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & \frac{4}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{11}{3} & -\frac{1}{3} & \frac{8}{3} & 1 \end{bmatrix} \xrightarrow{\frac{3}{11}R_3} \begin{bmatrix} 1 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & \frac{4}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{11} & \frac{8}{11} & \frac{3}{11} \end{bmatrix} \xrightarrow[2R_3+R_1]{-\frac{4}{3}R_3+R_2} \begin{bmatrix} 1 & 0 & 0 & -\frac{2}{11} & \frac{5}{11} \\ 0 & 1 & 0 & \frac{5}{11} & -\frac{7}{11} \\ 0 & 0 & 1 & -\frac{1}{11} & \frac{8}{11} \end{bmatrix}$$

so

$$A^{-1} = \begin{bmatrix} -\frac{2}{11} & \frac{5}{11} & \frac{6}{11} \\ \frac{5}{11} & -\frac{7}{11} & -\frac{4}{11} \\ -\frac{1}{11} & \frac{8}{11} & \frac{3}{11} \end{bmatrix}$$

SNB check: $$\begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} -\frac{2}{11} & \frac{5}{11} & \frac{6}{11} \\ \frac{5}{11} & -\frac{7}{11} & -\frac{4}{11} \\ -\frac{1}{11} & \frac{8}{11} & \frac{3}{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 5

a) **(13 points)**

Evaluate the surface integral

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iint_S \vec{F}\cdot d\vec{S}$$

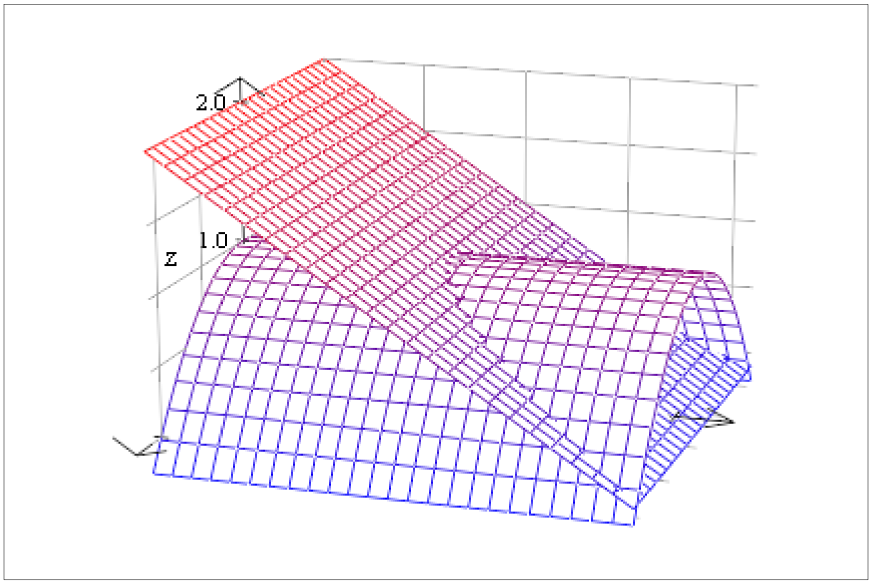
where

$$\vec{F} = xy\vec{i} + \left(y^2 + e^{xz^2}\right)\vec{j} + \sin(xy)\vec{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$, and the planes $z = 0, y = 0$, and $y + z = 2$. Sketch E.

Solution:

$1 - x^2$



We use the Divergence theorem.

$$\operatorname{div}\vec{F} = y + 2y = 3y$$

The parabolic cylinder intersects the x, y –plane at $x = \pm 1$. Thus $-1 \le x \le 1, 0 \le z \le 1 - x^2$, and $0 \le y \le 2 - z$. Therefore

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint_S \vec{F} \cdot d\vec{S} = \iiint_E 3y dV$$

$$= \int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{2-z} 3y dy dz dx$$

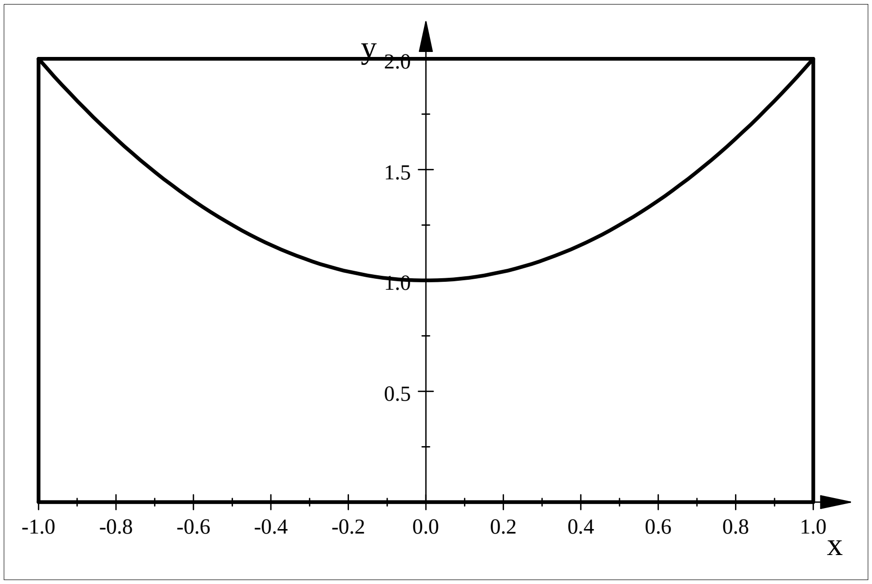
$$= 3\int_{-1}^{1} \int_{0}^{1-x^2} \frac{(2-z)^2}{2} dz dx$$

$$= \frac{3}{2}\int_{-1}^{1} \left[-\frac{(2-z)^3}{3} \right]_0^{1-x^2} dx$$

$$= -\frac{1}{2}\int_{-1}^{1} \left[\left(2 - 1 + x^2\right)^3 - 8 \right] dx$$

$$= -\frac{1}{2}\int_{-1}^{1} \left[\left(x^2 + 1\right)^3 - 8 \right] dx = -\frac{1}{2}\int_{-1}^{1} \left(x^6 + 3x^4 + 3x^2 - 7\right) dx = \frac{184}{35}$$

Or we can do the z integration first and then the y integration. The cylinder and the plane intersect when $1 - x^2 = 2 - y$, that is when $y = 1 + x^2$.



Thus

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint_S \vec{F} \cdot d\vec{S} = \iiint_E 3y dV$$
$$= \int_{-1}^{1} \int_0^{1+x^2} \int_0^{1-x^2} 3y dz dy dx + \int_{-1}^{1} \int_{1+x^2}^{2} \int_0^{2-y} 3y dz dy dx$$
$$= \frac{184}{35}$$

b) (**6 points**)

Let r be an eigenvalue for the constant square matrix A with corresponding eigenvector u. Show that $x(t) = t^r u$ is a solution of

$$tx'(t) = Ax(t) \quad t > 0$$

Solution: Since u is an eigenvector corresponding to r, then $Au = ru$. Thus

$$tx'(t) = rt\left(t^{r-1}\right)u = t^r(ru) = t^r Au = A(t^r u) = Ax(t)$$

so $t^r u$ is a solution of the DE.

5c) (**6 points**)

Show that if $t^r u$ is a solution of $tx'(t) = Ax(t) \quad t > 0$, then r be an eigenvalue for the constant square matrix A with corresponding eigenvector u.

Solution: We have from the DE that

$$t\left(rt^{r-1}u\right) = A(t^r u)$$

or

$$ru = Au$$

so u is an eigenvector corresponding to r.

Problem 6

a) (**12 points**)

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix}$$

$$\begin{vmatrix} 2-r & 3 \\ -1 & 6-r \end{vmatrix} = (2-r)(6-r)+3 = r^2 - 8r + 15 = (r-5)(r-3)$$

Therefore the eigenvalues are $3, 5$. The system of equations that yields the eigenvectors is

$$\begin{aligned} (2-r)x_1 + 3x_2 &= 0 \\ -x_1 + (6-r)x_2 &= 0 \end{aligned}$$

For $r = 3$ we have

$$\begin{aligned} -x_1 + 3x_2 &= 0 \\ -x_1 + 3x_2 &= 0 \end{aligned}$$

Thus $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector. For $r = 5$ we have

$$\begin{aligned} -3x_1 + 3x_2 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}$$

so $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector.

$\begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix}$, eigenvectors: $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \leftrightarrow 3, \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \leftrightarrow 5$

b) (**13 points**)

Solve the nonhomogeneous system of equations

$$x'(t) = Ax(t) + \begin{bmatrix} -e^t \\ e^{-t} \end{bmatrix}$$

where A is the matrix above in 6a).

Solution: The homogeneous solution is

$$x_h(t) = c_1 e^{3t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let

$$x_p(t) = \begin{bmatrix} ae^t + be^{-t} \\ ce^t + de^{-t} \end{bmatrix}$$

Then the DE

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -e^t \\ e^{-t} \end{bmatrix}$$

implies

$$\begin{bmatrix} ae^t - be^{-t} \\ ce^t - de^{-t} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix}\begin{bmatrix} ae^t + be^{-t} \\ ce^t + de^{-t} \end{bmatrix} + \begin{bmatrix} -e^t \\ e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} ae^t - be^{-t} \\ ce^t - de^{-t} \end{bmatrix} = \begin{bmatrix} 2ae^t + 2be^{-t} + 3ce^t + 3de^{-t} - e^t \\ -ae^t - be^{-t} + 6ce^t + 6de^{-t} + e^{-t} \end{bmatrix}$$

:

:Thus

$$\begin{aligned} a &= 2a + 3c - 1 \\ -b &= 2b + 3d \\ c &= -a + 6c \\ -d &= -b + 6d + 1 \end{aligned}$$

, Solution is: $\left[a = \frac{5}{8}, b = \frac{1}{8}, c = \frac{1}{8}, d = -\frac{1}{8}\right]$, From equation 3 we have $a = 5c.$ Equation 1 implies $2\,a + 3c = 1$ so we have $c = \frac{1}{8}$. Thus $a = \frac{5}{8}.$ From equation 2 we have $b = -d,$ so equation 4 implies $d = -\frac{1}{8}$ and $b = \frac{1}{8}$.

Thus

$$x_p(t) = \begin{bmatrix} ae^t + be^{-t} \\ ce^t + de^{-t} \end{bmatrix} = \begin{bmatrix} \frac{5}{8}e^t + \frac{1}{8}e^{-t} \\ \frac{1}{8}e^t - \frac{1}{8}e^{-t} \end{bmatrix}$$

Thus

$$x(t) = x_h(t) + x_p(t) = c_1 e^{3t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{5}{8}e^t + \frac{1}{8}e^{-t} \\ \frac{1}{8}e^t - \frac{1}{8}e^{-t} \end{bmatrix}$$

Check: $x_p(t) = \begin{bmatrix} \frac{5}{8}e^t + \frac{1}{8}e^{-t} \\ \frac{1}{8}e^t - \frac{1}{8}e^{-t} \end{bmatrix}$

$x_p'(t) = \begin{bmatrix} \frac{5}{8}e^t - \frac{1}{8}e^{-t} \\ \frac{1}{8}e^t + \frac{1}{8}e^{-t} \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix} x_p(t) + \begin{bmatrix} -e^t \\ e^{-t} \end{bmatrix} = \begin{bmatrix} \frac{5}{8}e^t - \frac{1}{8}e^{-t} \\ \frac{1}{8}e^t + \frac{1}{8}e^{-t} \end{bmatrix}$$

Problem 7

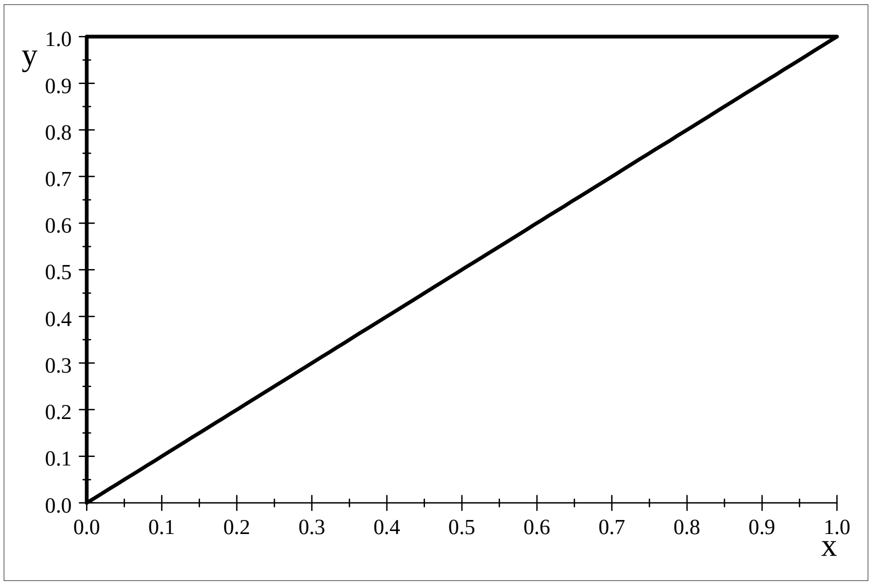
a) (12 points)

Evaluate the integral

$$\int_0^1 \int_x^1 \sin\left(y^2\right) dy dx$$

Sketch the region of integration.

x



Solution: We cannot evaluate the integral as it stands. We change the order of integration to get

$$\int_0^1 \int_0^y \sin\left(y^2\right) dx dy = \int_0^1 y \sin\left(y^2\right) dx = -\frac{1}{2} \cos\left(y^2\right) \Big]_0^1 = \frac{1}{2}(1 - \cos 1)$$

b) (**13 points**)

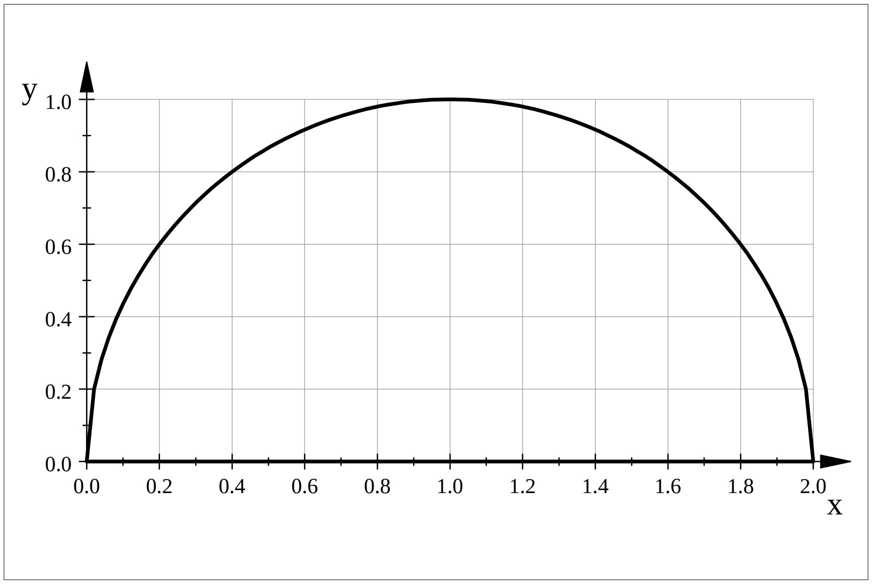
Evaluate

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy dx$$

Sketch the region of integration.

Solution: Given the form of the integral we switch to polar coordinates. The upper limit $y = \sqrt{2x - x^2}$ is the circle $2x - x^2 = y^2$ or $(x-1)^2 + y^2 = 1$. The region is the upper half of this circle, since y starts at 0. Thus the circle in polar coordinates is given $r = 2\cos\theta$.

$\sqrt{2x - x^2}$



Therefore

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dydx = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} (r) r dr d\theta = \int_0^{\frac{\pi}{2}} \frac{r^3}{3} \Big]_0^{2\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{8}{3} \cos^3\theta \right) d\theta$$

$$= \frac{8}{3} \left[\frac{3}{4} \sin\theta + \frac{1}{12} \sin 3\theta \right]_0^{\frac{\pi}{2}}$$

$$= \frac{8}{3} \left(\frac{3}{4} - \frac{1}{12} \right) = \frac{16}{9}$$

Problem 8

a) (13 points)

For what values of $c_1, c_2,$ and c_3 does the system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= c_1 \\ 3x_1 - x_2 + 2x_3 - 2x_4 &= c_2 \\ 5x_1 + 3x_2 - 4x_3 \quad\quad &= c_3 \end{aligned}$$

have a solution?

Solution: We form $\begin{bmatrix} 1 & 2 & -3 & 1 & c_1 \\ 3 & -1 & 2 & -2 & c_2 \\ 5 & 3 & -4 & 0 & c_3 \end{bmatrix}$ and row reduce.

$$\begin{bmatrix} 1 & 2 & -3 & 1 & c_1 \\ 3 & -1 & 2 & -2 & c_2 \\ 5 & 3 & -4 & 0 & c_3 \end{bmatrix} \xrightarrow[-5R_1+R_3]{-3R_1+R_2} \begin{bmatrix} 1 & 2 & -3 & 1 & c_1 \\ 0 & -7 & 11 & -5 & c_2 - 3c_1 \\ 0 & -7 & 11 & -5 & c_3 - 5c_1 \end{bmatrix}$$

$$\rightarrow_{-R_2+R_3} \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & c_1 \\ 0 & -7 & 11 & -5 & c_2 - 3c_1 \\ 0 & 0 & 0 & 0 & c_3 - 5c_1 - c_2 + 3c_1 \end{array}\right]$$

Therefore there is a solution for if

$$c_3 - 2c_1 - c_2 = 0$$

b) (**12 points**)

Express the initial value problem

$$y''' + ty'' + y' + 3y = e^t \sin t \qquad y(0) = 1, y'(0) = 2, y''(0) = 0$$

as a system of first order differential equation in normal form with an initial condition.

Solution: Let

$$x_1(t) = y(t),\ x_2(t) = y'(t), \ldots, x_n(t) = y^{(n-1)}(t)$$

Then

$$x_1'(t) = y'(t) = x_2(t)$$
$$x_2'(t) = y''(t) = x_3(t)$$
$$x_3'(t) = y'''(t) = -ty'' - y' - 3y + e^t \sin t$$
$$\text{or } x_3'(t) = -3x_1(t) - x_2(t) - tx_3(t) + e^t \sin t$$

Therefore the system is

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -1 & -t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^t \sin t \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \sin^3 x dx = \frac{1}{12}\cos 3x - \frac{3}{4}\cos x + C$
$\int \cos^3 x dx = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + C$
$\int \sec^2 \theta d\theta = \tan\theta + C$