Ma 227 **Final Exam Solutions** **5/15/06**

Name: ________________________________

Lecture Section: ________________________________

I pledge my honor that I have abided by the Stevens Honor System.

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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Directions: Answer all questions. The point value of each problem is indicated. If you need more work space, continue the problem you are doing on the **other side of the page it is on**.

There is a table of integrals at the end of the exam.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

#5 ________

#6 ________

#7 ________

#8 ________

Total ________

Problem 1

a) (**16 points**)

Compute the divergence and curl of the vector field

$$\vec{F}(x,y,z) = \left(y^2 + ze^{xz}\right)\vec{i} + 2xy\vec{j} + xe^{xz}\vec{k}$$

Solution:

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\left(y^2 + ze^{xz}\right)\vec{i} + 2xy\vec{j} + xe^{xz}\vec{k}\right) = z^2e^{xz} + 2x + x^2e^{xz}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + ze^{xz} & 2xy & xe^{xz} \end{vmatrix}$$

$$= \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + ze^{xz} & 2xy & xe^{xz} \end{matrix}\right| \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y^2 + ze^{xz} & 2xy \end{matrix}$$

$$= \left(\frac{\partial(xe^{xz})}{\partial y} - \frac{\partial 2xy}{\partial z}\right)\vec{i} + \left(\frac{\partial\left(y^2 + ze^{xz}\right)}{\partial z} - \frac{\partial(xe^{xz})}{\partial x}\right)\vec{j} + \left(\frac{\partial(2xy)}{\partial x} - \frac{\partial\left(y^2 + ze^{xz}\right)}{\partial y}\right)\vec{k}$$

$$= 0\vec{i} + (e^{xz} + xze^{xz} - e^{xz} - xze^{xz})\vec{j} + (2y - 2y)\vec{k} = 0$$

b) (**9 points**)

Does there exist a function $\phi(x,y,z)$ such that $\nabla\phi = \vec{F}$ where $\vec{F}$ is the vector field in 1a above. Why or why not? If yes. then find $\phi(x,y,z)$

Solution: Since curl$\vec{F} = 0$, $\vec{F}$ is a conservative force field and such a $\phi(x,y,z)$ does exist.

$$\phi_x = y^2 + ze^{xz}$$

so

$$\phi = xy^2 + e^{xz} + g(y,z)$$

so

$$\phi_y = 2xy + g_y = 2xy$$

Thus $g(y.z) = h(z)$ and

$$\phi = xy^2 + e^{xz} + h(z)$$

Then

$$\phi_z = xe^{xz} + h'(z) = xe^{xz}$$

so $h(z) = C,$ a constant. Thus

$$\phi = xy^2 + e^{xz} + C$$

Problem 2

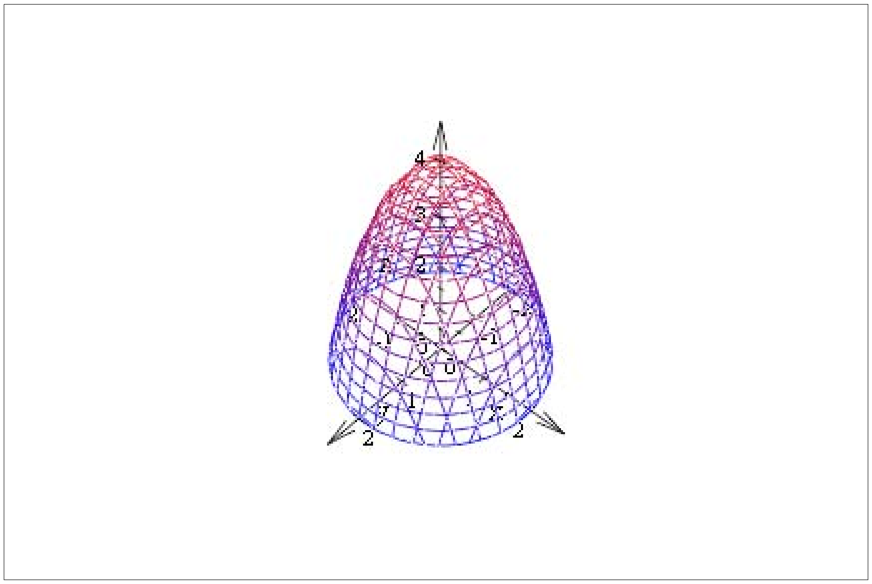
a) (**12 points**)

Give an expression in ploar coordinates for

$$\iint_S f(x,y,z)\,dS$$

where S is surface of the paraboloid $z = 4 - x^2 - y^2$ above the x,y –plane and $f(x,y,z) = x^2 + 2y^2 + z^2 - 4$. Sketch S. Do not evaluate the expression.

Solution: $z = 4 - x^2 - y^2$



Since the surface projects uniquely onto the x,y –plane and projects onto the x,y –plane in the circle $x^2 + y^2 \le 4$, then

$$\iint_S f(x,y,z)\,dS = \iint_S \left(x^2 + y^2 + z^2 - 4\right)\sqrt{1 + z_x^2 + z_y^2}\,dS$$

$$= \iint_{x^2+y^2\le 4} \left(x^2 + 2y^2 + \left(4 - x^2 - y^2\right)^2 - 4\right)\sqrt{1 + (-2x)^2 + (-2y)^2}\,dA$$

$$= \iint_{x^2+y^2\le 4} \left(x^2 + 2y^2 + \left(4 - x^2 - y^2\right)^2 - 4\right)\sqrt{1 + 4x^2 + 4y^2}\,dA$$

$$= \iint_{x^2+y^2\le 4} \left(x^4 + 2x^2y^2 - 7x^2 + y^4 - 6y^2 + 12\right)\sqrt{1 + 4x^2 + 4y^2}\,dA$$

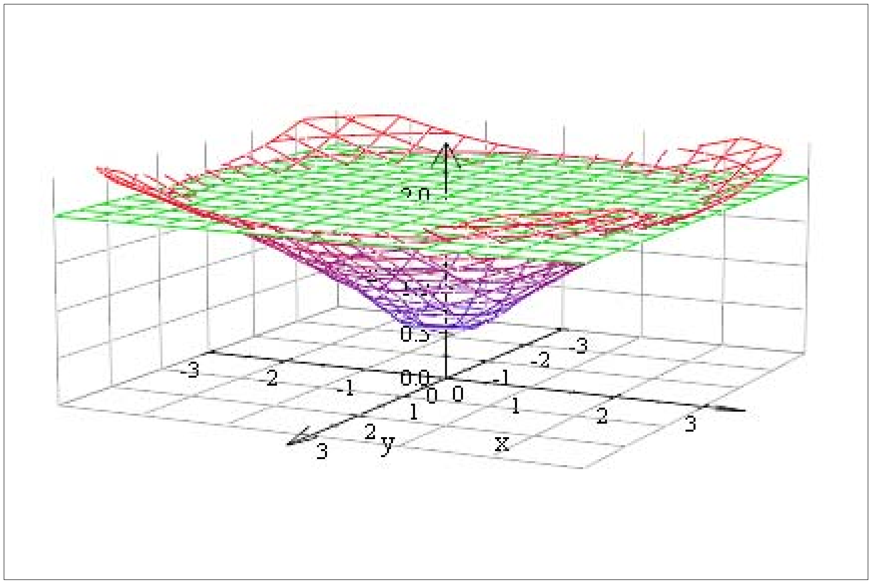
$$= \int_0^{2\pi}\int_0^2 \left(r^4\cos^4\theta + 2r^4\sin^2\theta\cos^2\theta - 7r^2\cos^2\theta + r^4\sin^4\theta - 6r^2\sin^2\theta + 12\right) r\sqrt{1 + 4r^2}\,dr\,d\theta$$

Use Stokes' Theorem to evaluate $\iint_S \left(\nabla \times \vec{F}\right)\cdot \vec{n}\,dS$ where $\vec{F}(x,y,z) = y\vec{i} + y\vec{j} + z\vec{k}$ and S is the surface defined by $S = \left\{z = \left(x^2 + y^2\right)^{\frac{1}{3}},\ 0 \le z \le 2\right\}$ Assume that S is oriented upwards.

Solution:

$$\iint_S \left(\nabla \times \vec{F}\right)\cdot \vec{n}\,dS = \oint_C \vec{F}\cdot d\vec{r}$$

where C is the boundary of the surface S. Below is the surface.



The boundary of the surface is $\left(x^2 + y^2\right)^{\frac{1}{3}} = 2$ which is the circle $x^2 + y^2 = 8,\ z = 2.$ We parametrize this as

$$\vec r(t) = \sqrt{8}\,\cos t\vec i + \sqrt{8}\,\sin t\vec j + 2\vec k, \quad 0 \le t \le 2\pi$$

Then

$$\vec r\,'(t) = -\sqrt{8}\,\sin t\vec i + \sqrt{8}\,\cos t\vec j$$
$$\vec F(t) = \sqrt{8}\,\sin t\vec i + \sqrt{8}\,\sin t\vec j = 2\vec k$$

so

$$\iint_S \left(\nabla \times \vec F\right) \cdot \vec n dS = \oint_C \vec F \cdot d\vec r = \int_0^{2\pi} \left(-8\sin^2 t + 8\sin t\cos t\right)dt$$
$$= \left[4\cos t\sin t - 4t + 4\sin^2 t\right]_0^{2\pi} = -8\pi$$

Problem 3

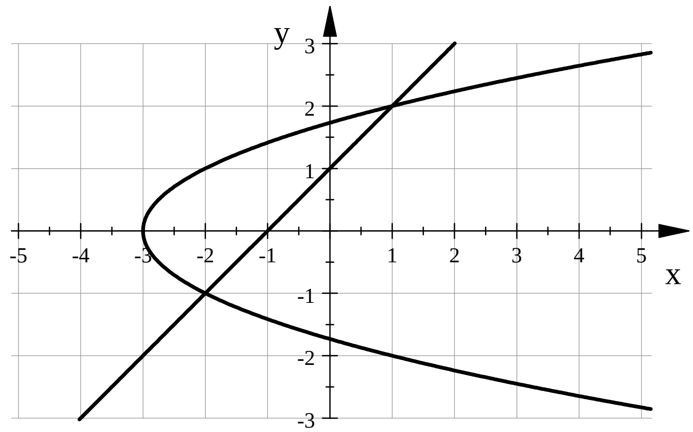
a) (12 points)

Reverse the order of integration:

$$\int_{-1}^{2}\int_{y^2-3}^{y-1}(x^2 + xy)dxdy$$

Be sure to sketch the region of integration. Do *not* evaluate.

Solution: The region of integration looks like:



$x = y^2 - 3$

If we equate $y - 1 = y^2 - 3$, then,

$\Rightarrow y^2 - y - 2 = 0$

$\Rightarrow (y-2)(y+1) = 0$

$\Rightarrow y = -1$ and $y = 2$.

If $y = -1$, $x = -2$ and if $y = 2$, $x = 1$.

We need to split to two double integrals. The final solution is:

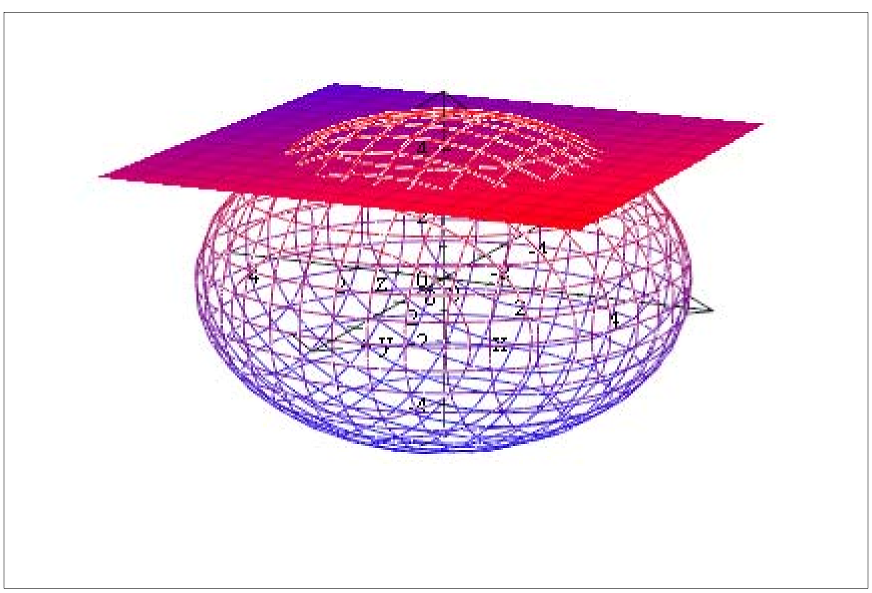
$$\int_{-3}^{-2}\int_{-\sqrt{x+3}}^{\sqrt{x+3}} (x^2 + xy)dydx + \int_{-2}^{1}\int_{x+1}^{\sqrt{x+3}} (x^2 + xy)dydx$$

b) (**13 points**)

Set up a triple integral in spherical coordinates to find the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 25$ and below by the plane $z = 4$. Do *not* evaluate

Solution: The region of integration is shown below.

$x^2 + y^2 + z^2 = 25$



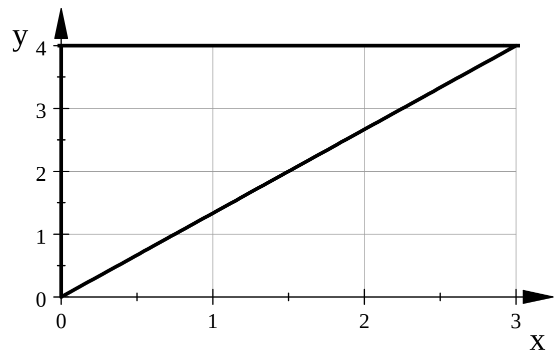
ρ will go from the plane $z = 4$ to the sphere $x^2 + y^2 + z^2 = 25$.

In spherical, $x^2 + y^2 + z^2 = 25 \Rightarrow \rho = 5$

Also, $z = 4 \Rightarrow \rho\cos\phi = 4 \Rightarrow \rho = 4\sec\phi.$

So, $4\sec\phi \le \rho \le 5.$

For ϕ, we can form a right triangle with hypotenuse 5 (the radius of the sphere) and vertical side 4 which is the distance from the origin to $z = 4$. So the horizontal side is 3.



$\frac{4}{3}x$

Therefore, $\tan\phi = \frac{3}{4} \Rightarrow \phi = \arctan(\frac{3}{4}).$

So, $0 \le \phi \le \arctan(\frac{3}{4}).$

The volume is:

$$V = \int_0^{2\pi}\int_0^{\arctan(\frac{3}{4})}\int_{4\sec\phi}^{5} p^2 \sin\phi d\rho d\phi d\theta$$

Problem 4

a) (**15 points**)

Evaluate the line integral

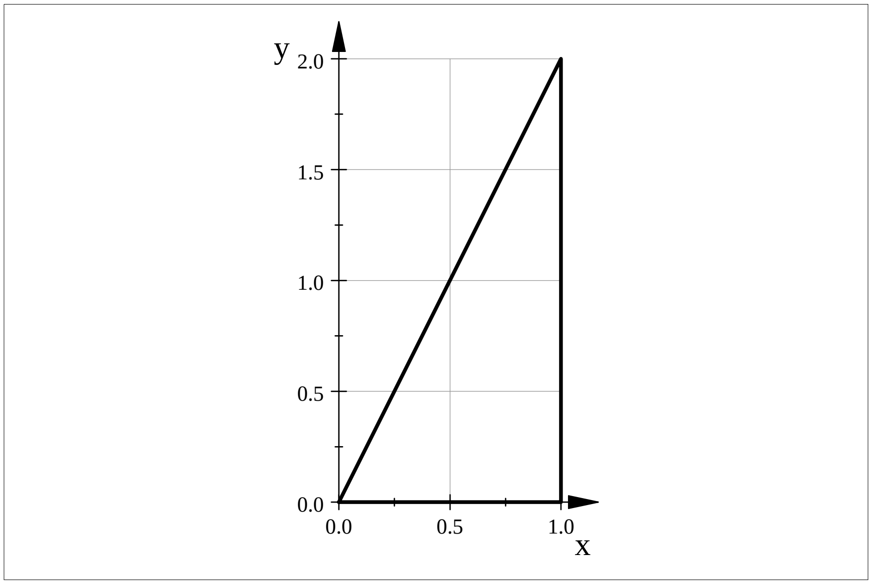
$$\oint_C \left(xy + \ln\left(1 + x^2\right)\right)dx + xdy$$

where C is the triangle with vertices $(0,0), (1,0), (1,2)$ oriented in a counterclockwise manner. Sketch C.

Solution: We use Green's Theorem

$$\oint_C Pdx + Qdy = \iint_R (Q_x - P_y)dA$$

$(0,0,1,0,1,2,0,0)$



$P = xy + \ln(1 + x^2)$ and $Q = x$ so that $P_y = x$ and $Q_x = 1$. The line joining $(0,0)$ to $(1,2)$ is $y = 2x$. Thus

$$\oint_C P dx + Q dy = \iint_R (Q_x - P_y) dA = \iint_{Triangle} (1-x) dA$$
$$= \int_0^2 \int_{\frac{y}{2}}^1 (1-x) dx dy$$
$$= \int_0^1 \int_0^{2x} (1-x) dy dx = \frac{1}{3}$$

b) (**10 points**)

Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \to_{-R_1+R_2} \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix} \to_{3R_2+R_1} \begin{bmatrix} 2 & 0 & -2 & 3 \\ 0 & -1 & -1 & 1 \end{bmatrix} \to_{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & -1 & \frac{3}{2} \\ 0 & -1 & -1 & 1 \end{bmatrix}$$
$$\to_{-R_2} \begin{bmatrix} 1 & 0 & -1 & \frac{3}{2} \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix}$

$\begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$, inverse: $\begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix}$

Problem 5

a) (**13 points**)

Evaluate the surface integral

$$\iint_S \vec{F} \cdot \vec{n} dS$$

where

$$\vec{F} = \left(2x^2 y\vec{i} + 6y^2 z\vec{j} + 2xz\vec{k}\right)$$

and S is the surface bounded by the unit cube $S = \{0 \le x, y, z \le 1\}$.

Solution:

$$\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV$$

$$\nabla \cdot \vec{F} = 4xy + 12yz + 2x$$

so

$$\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV = \int_0^1 \int_0^1 \int_0^1 (4xy + 12yz + 2x) dx dy dz$$

$$= \int_0^1 \int_0^1 (2y + 12yz + 1) dy dz = \int_0^1 (1 + 6z + 1) dz = 2 + 3 = 5$$

b) (**7 points**)

Let

$$A = \begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix}$$

Find the characteristic polynomial $p(r)$ for A.

Solution:

$$\begin{vmatrix} 2-r & 5 \\ 1 & -3-r \end{vmatrix} = -(2-r)(3+r) - 5 = r^2 + r - 11$$

c) (**5 points**)

Show that $p(A) = 0,$ where A is the matrix in 5b) and $p(r)$ is the characteristic polynomial of A.

Solution:

$$p(A) = \begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix} - 11\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -5 \\ -1 & 14 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 6

a) (**12 points**)
Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Solution:
$\det\begin{bmatrix} 1-r & 2 \\ 2 & 1-r \end{bmatrix} = (1-r)^2 - 4 = r^2 - 2r - 3 = (r-3)(r+1).$ Thus the eigenvalues are $r = 3, -1$.
The system of equations for the eigenvectors is

$$\begin{aligned} (1-r)x_1 + 2x_2 &= 0 \\ 2x_1 + (1-r)x_2 &= 0 \end{aligned}$$

Setting $r = 3$ leads to $x_1 = x_2$ so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector.

Setting $r = -1$ leads to $x_1 = -x_2$ so $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector.

b) (**13 points**)
Solve the nonhomgeneous system of equations

$$x'(t) = Ax(t) + \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$$

where A is the matrix above in 6a).
Soltuion:

$$y_h = c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Let

$$y_p = e^{2t}\begin{bmatrix} a \\ b \end{bmatrix}$$

Then the DE implies

$$e^{2t}\begin{bmatrix} 2a \\ 2b \end{bmatrix} = e^{2t}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} + e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 2a \\ 2b \end{bmatrix} = \begin{bmatrix} a + 2b + 1 \\ 2a + b \end{bmatrix}$$

We therefore have the system

$$a - 2b = 1$$
$$-2a + b = 0$$

Hence $a = -\frac{1}{3}, b = -\frac{2}{3}$. Hence

$$y_p = e^{2t}\begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$$

and

$$y_g = y_h + y_g = c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t}\begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$$

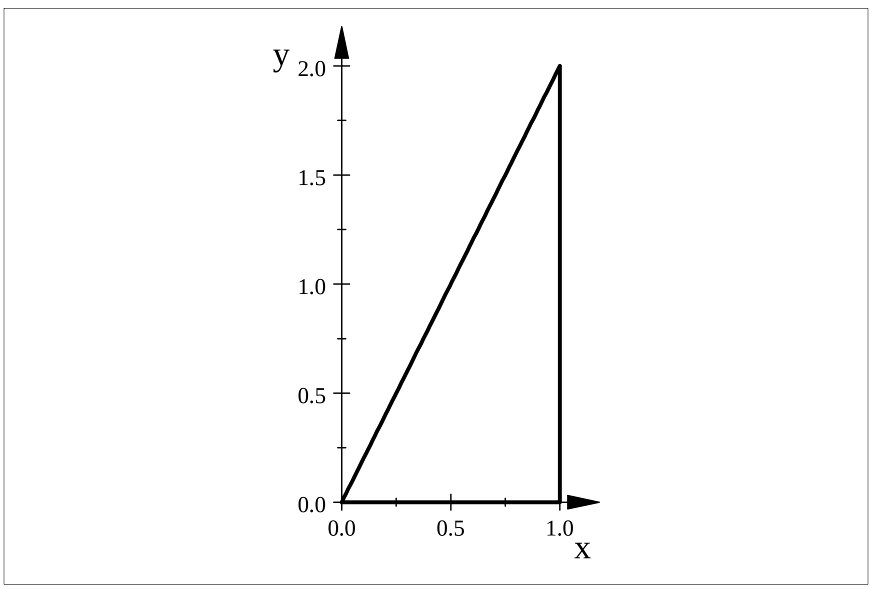
Problem 7

a) (**12 points**)
Evaluate the integral

$$\int_0^2 \int_{\frac{y}{2}}^1 \sqrt{6 - x^2}\, dxdy$$

Sketch the region of integration.

Solution: The sketch of the region looks like the following:
$y = 2x$



We will need to reverse the order of integration to evaluate the integral.

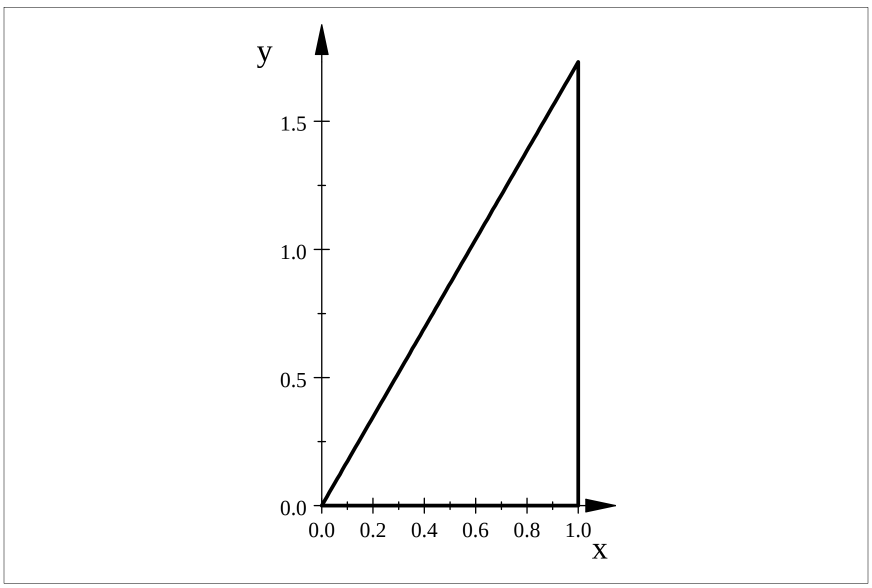
$$\int_0^1 \int_0^{2x} \sqrt{6-x^2}\, dydx$$
$$= \int_0^1 2x\sqrt{6-x^2}\, dx$$
$$u = 6 - x^2, du = -2xdx$$
$$= \int -\sqrt{u}\, du = -\frac{2}{3}(6-x^2)^{\frac{3}{2}}|_0^1 = -\frac{2}{3}(5)^{\frac{3}{2}} + \frac{2}{3}(6)^{\frac{3}{2}}$$

b) (**13 points**)
Convert to Polar Coordinates and evaluate

$$\int_0^1 \int_0^{\sqrt{3}x} 3xdydx$$

Sketch the region of integration.

Solution: The region of integration is a right triangle:
$\sqrt{3}\,x$
$\sqrt{3} = 1.7321$



In polar coordinates, r goes from 0 to the line $x = 1$.

$$x = 1 \Rightarrow r\cos\theta = 1 \Rightarrow r = \sec\theta.$$

From the triangle, $\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$. This is the upper limit on θ.
Therefore,

$$\begin{aligned}\int_0^1\int_0^{\sqrt{3}\,x} 3xdydx &= \int_0^{\frac{\pi}{3}}\int_0^{\sec\theta} 3r\cos\theta rdrd\theta \\ &= \int_0^{\frac{\pi}{3}} r^3\cos\theta|_0^{\sec\theta}d\theta \\ &= \int_0^{\frac{\pi}{3}} \sec^3\theta\cos\theta d\theta \\ &= \int_0^{\frac{\pi}{3}} \sec^2\theta d\theta = \tan\theta|_0^{\pi/3} = \tan(\frac{\pi}{3}) = \sqrt{3}\end{aligned}$$

Problem 8

a) (**13 points**)

Solve the system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 0 \\ 3x_1 - x_2 + 2x_3 - 2x_4 &= 4 \\ 5x_1 + 3x_2 - 4x_3 \qquad &= 4 \end{aligned}$$

Solution: We form the augmented matrix

$$\begin{bmatrix} 1 & 2 & -3 & 1 & 0 \\ 3 & -1 & 2 & -2 & 4 \\ 5 & 3 & -4 & 0 & 4 \end{bmatrix}$$

and row reduce it to reduced echelon form.

$$\begin{bmatrix} 1 & 2 & -3 & 1 & 0 \\ 3 & -1 & 2 & -2 & 4 \\ 5 & 3 & -4 & 0 & 4 \end{bmatrix} \to_{-3R_1+R_2;-5R_1+R_3} \begin{bmatrix} 1 & 2 & -3 & 1 & 0 \\ 0 & -7 & 11 & -5 & 4 \\ 0 & -7 & 11 & -5 & 4 \end{bmatrix}$$

$$\to_{-R_2+R_3;R_2/-7} \begin{bmatrix} 1 & 2 & -3 & 1 & 0 \\ 0 & 1 & -11/7 & 5/7 & -4/7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to_{-2R_2+R_1} \begin{bmatrix} 1 & 0 & 1/7 & -3/7 & 8/7 \\ 0 & 1 & -11/7 & 5/7 & -4/7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $x_1 = \frac{-x_3+3x_4+8}{7}, x_2 = \frac{11x_3-5x_4-4}{7}$ and x_3 and x_4 are arbitrary.

b) (**12 points**)

Give one differential equation that is equivalent to the system

$$x'(t) = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 3\sin(2t) \end{bmatrix}$$

Solution: The system is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 3\sin(2t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_1(t) + 3x_2(t) + 3\sin(2t) \end{bmatrix}$$

Thus

$$\begin{aligned} x_1' &= x_2 \\ x_1'' &= x_2' = x_1 + 3x_1' + 3\sin(2t) \end{aligned}$$

so the DE is

$$y'' - 3y' - y = 3\sin(2t)$$

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$
$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$
$\int \sec^2 \theta d\theta = \tan\theta + C$