You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Directions: Answer all questions. The point value of each problem is indicated. If you need more work space, continue the problem you are doing on the other side of the page it is on.

There is a table of integrals at the end of the exam.

Score on Problem #1 ______
#2 ______
#3 ______
#4 ______
#5 ______
#6 ______
#7 ______
#8 ______

Total ______

I pledge my honor that I have abided by the Stevens Honor System.

_________________________________________
Problem 1

a) (13 points)

Does the following system of equations have a unique solution or an infinite set of solutions or no solution? Find all solutions, if they exist.

\[
\begin{align*}
    x_1 + x_2 + 2x_3 + 2x_4 &= 6 \\
    x_1 + 2x_2 + 3x_3 - x_4 &= 5 \\
    2x_1 + 3x_2 + 5x_3 + x_4 &= 11 \\
    x_1 + 3x_2 + 4x_3 - 2x_4 &= 6
\end{align*}
\]
b) (12 points)  
Show that 

$$ \text{div(curl}(\mathbf{F})) = \nabla \cdot (\nabla \times \mathbf{F}) = \mathbf{0}. $$

(Assume that the mixed second partial derivatives are equal.) \( \mathbf{F}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}. \)
Problem 2

a) (10 points)
Find the eigenvalues and eigenvectors of

\[ A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}. \]
b) (15 points)
Find the [real] general solution to
\[
\begin{pmatrix}
  x_1' \\
  x_2'
\end{pmatrix} = \begin{pmatrix}
  2 & -1 \\
  1 & 2
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix} + \begin{pmatrix}
  0 \\
  12e^{2t}
\end{pmatrix}.
\]
Problem 3

a) (12 points)
Let $R$ be the region in the first octant bounded by the plane $x + 2y + 3z = 6$. Sketch the region $R$. Set up three iterated integrals for the volume with the orders of integration as specified below.

\[ \iiint dzdydx, \; \iiint dydxdz, \; \iiint dxdzdy \]
b) (13 points)
The integral below represents the volume of a solid. Describe the solid. Evaluate the integral.

\[ \int_{-2}^{2} \int_{-2}^{0} \int_{\sqrt{4-x^2-y^2}}^{-\sqrt{4-x^2-y^2}} dz
dy
dx \]
Problem 4

a) (10 points)

Evaluate

\[ \int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \mathbf{n} dS \]

where

\[ \mathbf{F} = x^2 z^3 \mathbf{i} + 2xyz^3 \mathbf{j} + xz^4 \mathbf{k} \]

and \( S \) is the surface of the box with vertices \((\pm 1, \pm 2, \pm 3)\).
b) (15 points)

Evaluate

$$\iint_{S} \vec{F} \cdot d\vec{S} = \iint_{S} \vec{F} \cdot \vec{n} dS$$

where

$$\vec{F} = x^2 \vec{i} + xy \vec{j} + z \vec{k}$$

and $S$ is the portion of the paraboloid $z = x^2 + y^2$ below $z = 1$, oriented with the normal pointing downward.
Problem 5
(25 points)

Verify Stokes’ Theorem for the vector field \( \mathbf{F} = -y \mathbf{i} + 2x \mathbf{j} + (z + x) \mathbf{k} \) over the upper hemisphere \( x^2 + y^2 + z^2 = 1, \ z \geq 0 \), oriented by the outward pointing normal \( \hat{n} \).
Problem 6

a) (13 points)

It can be shown that for the force field

\[ \mathbf{F}(x, y, z) = e^x \cos y \mathbf{i} - e^x \sin y \mathbf{j} + 2 \mathbf{k} \]

that

\[ \text{curl} \mathbf{F} = 0 \]

Find the work done by \( \mathbf{F} \) on an object moving along a curve \( C \) from \( (0, \frac{\pi}{2}, 1) \) to \( (1, \pi, 3) \).
6 b) (12 points)
Evaluate
\[ \oint_C \left( \arctan x + y^2 \right) dx + \left( e^y - x^2 \right) dy \]
where \( C \) is the path enclosing the annular region shown below, positively oriented.
Problem 7

a) (13 points)

Calculate

$$\iint_{R} (x^2 + y^2)^{-2} dA$$

where $R$ is the part of the circle centered at $(1, 0)$ of radius 1 to the right of the line $x = 1$ in the first quadrant. Be sure to sketch $R$. 
b) (12 points) 
Evaluate 

$$
\iiint_E zdV 
$$

where $E$ is the region within the cylinder $x^2 + y^2 = 4$, where $0 \leq z \leq y$. 
Problem 8

a) (13 points)
It can be shown that the Cauchy-Euler system

\[ tx' = Ax(t) \quad t > 0 \]

where \( A \) is a constant matrix, has a nontrivial solutions of the form

\[ x(t) = e^{rt}u \]

if and only if \( r \) is an eigenvalue of \( A \) and \( u \) is a corresponding eigenvector. Use this information to solve the system

\[ tx' = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} x(t) \quad t > 0 \]

b) (12 points)
Rewrite the system of equations

\[ x' = Ax + f \]

where

\[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad f = \begin{bmatrix} 0 \\ 0 \\ \cos t \end{bmatrix} \]

as a single scalar equation.
Table of Integrals

\[
\int \sin^2 x \, dx = -\frac{1}{2} \cos x \sin x + \frac{1}{2} x + C
\]
\[
\int \cos^2 x \, dx = \frac{1}{2} \cos x \sin x + \frac{1}{2} x + C
\]
\[
\int \sin^3 x \, dx = -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + C
\]
\[
\int \cos^3 x \, dx = \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C
\]
\[
\int (\cos^2 x - \sin^2 x) \, dt = \frac{1}{2} \sin 2x + C
\]