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Ma 227 Final Exam Solutions 12/22/09

Name: _________________________________ **ID**: __________________________

Lecture Section: _________________________________

Problem 1

a) **(13 points)**

Does the following system of equations have a unique solution or an infinite set of solutions or no solution? Find any solutions.

$$\begin{aligned} x_1 + x_2 + 2x_3 + 2x_4 &= 6 \\ x_1 + 2x_2 + 3x_3 - x_4 &= 5 \\ 2x_1 + 3x_2 + 5x_3 + x_4 &= 11 \\ x_1 + 3x_2 + 4x_3 - 2x_4 &= 6 \end{aligned}$$

Solution: We use row reduction on the augmented matrix.

$$\begin{bmatrix} 1 & 1 & 2 & 2 & 6 \\ 1 & 2 & 3 & -1 & 5 \\ 2 & 3 & 5 & 1 & 11 \\ 1 & 3 & 4 & -2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 2 & 6 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 2 & 2 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 2 & 6 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 2 & 2 & 6 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, with the reduced row echelon form, we can see that the system has an infinite number of solutions. x_3 can take any value.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

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b) **(12 points)**
Show that

$$\mathrm{div}(\mathrm{curl}(\vec{F})) = \nabla \cdot \left(\nabla \times \vec{F}\right) = \vec{0}.$$

(Assume that the mixed second partial derivatives are equal.) $\vec{F}(x,y,z) = P\vec{i} + Q\vec{j} + R\vec{k}.$

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Solution. Let

$$
\begin{aligned}
\vec{F} &= P\vec{i} + Q\vec{j} + R\vec{k} \\
\nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ P & Q & R \end{vmatrix} \\
&= (Ry - Q_z)\vec{i} - (R_x - P_z)\vec{j} + (Q_x - P_y) \\
\nabla \bullet \left(\nabla \times \ \vec{F}\right) &= R_{yx} - Q_{zx} - Rxy + P_{zy} + Q_{xz} - P_{yz} \\
&= \ \vec{0}
\end{aligned}
$$

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Problem 2

a) (10 points)

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

Solution: We solve $\det(A - rI) = 0$.

$$\begin{aligned} \det(A - rI) &= \begin{vmatrix} 2-r & -1 \\ 1 & 2-r \end{vmatrix} \\ &= (2-r)^2 + 1 \\ (2-r)^2 &= -1 \\ 2 - r &= \pm i \\ r &= 2 \pm i \end{aligned}$$

So, the eigenvalues are a complex conjugate pair. We find the eigenvector for one and take the complex conjugate to get the other. For $r = 2 + i,$ we solve

$$(A - rI)u = 0$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The second row is redundant, so $-iu_1 - u_2 = 0$ or $u_2 = -i \cdot u_1$. Hence any multiple of $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector for $r = 2 + i$. Then an eigenvector corresponding to $r = 2 - i$ is $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

b) (13 points)

Find the [real] general solution to

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 12e^{2t} \end{bmatrix}.$$

Solution: The solution is the general solution (x_h) to the homogeneous equation plus one [particular] solution (x_p) to the full non-homogeneous equation. First we'll find x_p. It is in the form

$$x_p = \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{2t} \end{bmatrix}$$

Substituting into the d.e., we obtain

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$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2c_1e^{2t} \\ 2c_2e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1e^{2t} \\ c_2e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} 2c_1e^{2t} - c_2e^{2t} \\ c_1e^{2t} + 2c_2e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} 2c_1e^{2t} \\ 2c_2e^{2t} \end{bmatrix} = \begin{bmatrix} 2c_1e^{2t} - c_2e^{2t} \\ c_1e^{2t} + 2c_2e^{2t} \end{bmatrix} + \begin{bmatrix} 0 \\ 12e^{2t} \end{bmatrix}$$

We can divide by e^{2t} (which is never zero) and move the unknowns to the left side to obtain

$$\begin{bmatrix} c_2 \\ -c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \end{bmatrix}$$

$$x_p = \begin{bmatrix} -12e^{2t} \\ 0 \end{bmatrix}$$

For the solution to the homogeneous equation, we use one of the eigenvalues and eigenvectors found in 2a to write a complex solution and break it into real and imaginary parts. we'll use $2 + i$.

$$x = e^{(2+i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{2t}(\cos t + i\sin t)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t}\cos t + ie^{2t}\sin t \\ e^{2t}\sin t - ie^{2t}\cos t \end{bmatrix}$$

$$x_h = c_1\begin{bmatrix} e^{2t}\cos t \\ e^{2t}\sin t \end{bmatrix} + c_2\begin{bmatrix} e^{2t}\sin t \\ -e^{2t}\cos t \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t}\cos t & e^{2t}\sin t \\ e^{2t}\sin t & -e^{2t}\cos t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Finally, we add to obtain the desired solution.

$$x = \begin{bmatrix} e^{2t}\cos t & e^{2t}\sin t \\ e^{2t}\sin t & -e^{2t}\cos t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -12e^{2t} \\ 0 \end{bmatrix} \backslash$$

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Problem 3

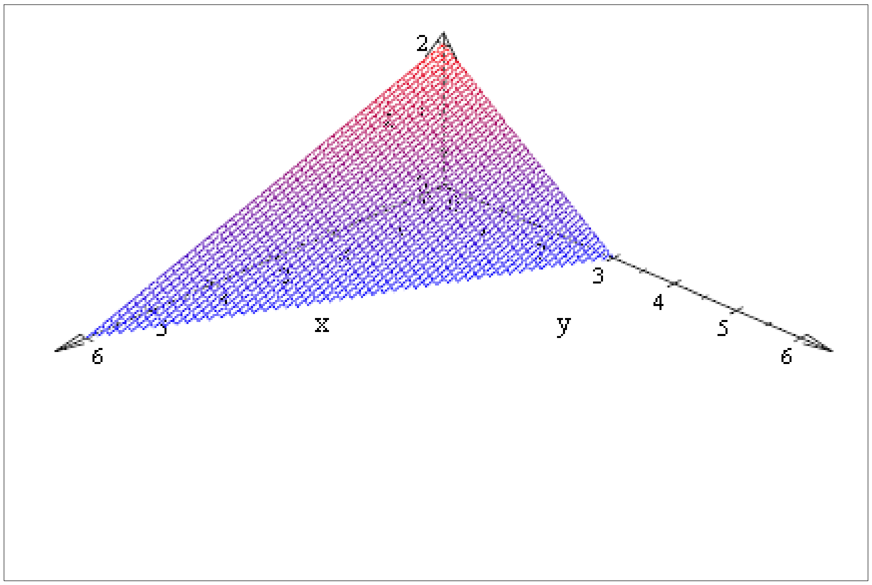
a) (12 points)

Let R be the region in the first octant bounded by the plane $x + 2y + 3z = 6$. Sketch the region R. Set up three iterated integrals for the volume with the orders of integration as specified below.

$$\iiint dzdydx \quad , \quad \iiint dydxdz \quad , \quad \iiint dxdzdy$$

Solution: The region is the tetrahedron bounded by the $x-y$ plane, the $x-z$ plane, the $y-z$ plane and the plane given.

$\frac{1}{3}(6-x-2y)$



The integrals are

$$V = \int_0^6 \int_0^{3-\frac{1}{2}x} \int_0^{2-\frac{1}{3}x-\frac{2}{3}y} dzdydx = \int_0^2 \int_0^{6-3z} \int_0^{3-\frac{1}{2}x-\frac{3}{2}z} dydxdz = \int_0^3 \int_0^{2-\frac{2}{3}y} \int_0^{6-2y-3z} dxdzdy$$

b) (13 points)

The integral below represents the volume of a solid. Describe the solid. Evaluate the integral.

$$\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dzdydx$$

Solution: The region is a hemisphere - the right half of a sphere of radius 2 centered at the origin and bounded by the $x-z$ plane (with y positive.) The integral is easily evaluated with a change to spherical coordinates.

$$\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dzdydx = \int_0^{\pi} \int_0^{\pi} \int_0^{2} \rho^2 \sin\phi d\rho d\phi d\theta$$

$$= \int_0^{\pi} \int_0^{\pi} \frac{2^3}{3} \sin\phi d\phi d\theta = \frac{8}{3} \int_0^{\pi} \cos\phi|_{\phi=0}^{\phi=\pi} d\theta$$

$$= \frac{8}{3} \cdot 2 \int_0^{\pi} d\theta = \frac{16}{3}\pi$$

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Problem 4

a) (**10 points**)

Evaluate

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$$

where

$$\vec{F} = x^2z^3\,\vec{i} + 2xyz^3\,\vec{j} + xz^4\,\vec{k}$$

and S is the surface of the box with vertices $(\pm1, \pm2, \pm3)$.

Solution: We use the divergence theorem to replace the surface integral with something that [we hope] is simpler.

$$\iiint_{Box} \nabla \cdot \vec{F} dV = \iint_S \vec{F} \cdot d\vec{S}$$

$$\nabla \cdot \vec{F} = 2xz^3 + 2xz^3 + 4xz^3 = 8xz^3$$

$$\iiint_{Box} \nabla \cdot \vec{F} dV = \int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} 8xz^3 dzdydx$$

Since the inner integral yields

$$\int_{-3}^{3} 8xz^3 dz = 2xz^4\Big|_{z=-3}^{z=3} = 8x\left(3^4 - (-3)^4\right) = 0$$

the result is zero.

b) (**15 points**)

Evaluate

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$$

where

$$\vec{F} = x^2\vec{i} + xy\vec{j} + z\vec{k}$$

and S is the portion of the paraboloid $z = x^2 + y^2$ below $z = 1$, oriented with the normal pointing downward.

Solution: The surface is simply parametrized using x and y.

$$\vec{r} = x\vec{i} + y\vec{j} + \left(x^2 + y^2\right)\vec{k}$$

$$\vec{r}_x = \vec{i} + 2x\vec{k}$$

$$\vec{r}_y = \vec{j} + 2y\vec{k}$$

$$\begin{aligned} \vec{r}_x \times \vec{r}_y &= \left(\vec{i} + 2x\vec{k}\right) \times \left(\vec{j} + 2y\vec{k}\right) \\ &= \left(\vec{i} \times \vec{j}\right) + 2y\left(\vec{i} \times \vec{k}\right) + 2x\left(\vec{k} \times \vec{j}\right) + 4xy\left(\vec{k} \times \vec{k}\right) \\ &= \vec{k} - 2y\vec{j} - 2x\vec{i} \end{aligned}$$

We observe that this is a normal pointing upward, but the other direction is required

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_y \times \vec{r}_x) dA_{xy}$$

$$= \iint_{x^2+y^2 \le 1} \left(2x^3 + 2xy^2 - \left(x^2 + y^2\right)\right) dA$$

$$= \int_0^{2\pi} \int_0^1 \left(2r^3 \cos\theta - r^2\right) r dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(2r^4 \cos\theta - r^3\right) r dr d\theta$$

$$\int_0^{2\pi} \left(\frac{2}{5} \cos\theta - \frac{1}{4}\right) d\theta$$

$$= -\frac{1}{2}\pi$$

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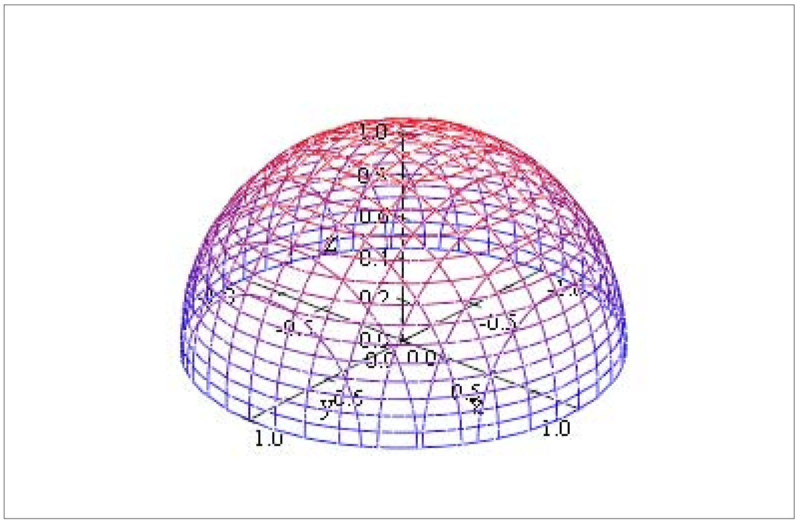
Problem 5

(25 points)

Verify Stokes' Theorem for the vector field $\vec{F} = -y\vec{i} + 2x\vec{j} + (z+x)\vec{k}$ over the upper hemisphere $x^2 + y^2 + z^2 = 1,\ z \geq 0,$ oriented by the outward pointing normal $\vec{n}$.

Solution: The hemisphere is shown below

$x^2 + y^2 + z^2 = 1$



Stokes' Theorem is

$$\int\int_S \operatorname{curl}\vec{F} \cdot \vec{n}ds = \int\int_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}ds = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

We first evaluate $\oint_{\partial S} \vec{F} \cdot d\vec{r}$. (This is worth 10 points.)

The boundary of S is the circle $x^2 + y^2 = 1,\ \ z = 0.$ We parametrize ∂S as $x = \cos t, y = \sin t, z = 0,$ $0 \leq t \leq 2\pi.$ Then

$$\vec{F}(t) = -\sin t\vec{i} + 2\cos t\vec{j} + (0 + \cos t)\vec{k}$$

$$\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + 0\vec{k}$$

$$\vec{r}'(t) = -\sin t\vec{i} + \cos t\vec{j} + 0\vec{k}$$

Hence

$$\vec{F}(t) \cdot \vec{r}'(t) = \sin^2 t + 2\cos^2 t$$

and

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(\sin^2 t + 2\cos^2 t\right) dt$$

$$= -\frac{1}{2}\cos t \sin t + \frac{1}{2}t + \cos t \sin t + t\Big|_0^{2\pi}$$

$$= \frac{1}{2}\cos t \sin t + \frac{3}{2}t\Big|_0^{2\pi} = 3\pi$$

We now evaluate

$$\int\int_S \operatorname{curl}\vec{F} \cdot \vec{n}ds$$

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(This is worth 15 points.)

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & 2x & x+z \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & 2x & x+z \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -y & 2x \end{matrix}$$

$$= 0\vec{i} + 0\vec{j} + 2\vec{k} + \vec{k} - 0\vec{i} - \vec{j} = -\vec{j} + 3\vec{k}$$

We use spherical coordinates to parametrize the hemisphere. Thus since $\rho = 1$ here

$$x = \cos\theta\sin\phi,\ \ y = \sin\theta\sin\phi,\ \ z = \cos\phi$$

Therefore

$$\vec{r}(\theta,\phi) = \cos\theta\sin\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\phi\vec{k}$$

$$\vec{r}_\theta = -\sin\theta\sin\phi\vec{i} + \cos\theta\sin\phi\vec{j}$$

$$\vec{r}_\phi = \cos\theta\cos\phi\vec{i} + \sin\theta\cos\phi\vec{j} - \sin\phi\vec{k}$$

$$\vec{N} = \vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ -\sin\theta\sin\phi & \cos\theta\sin\phi \\ \cos\theta\cos\phi & \sin\theta\cos\phi \end{matrix}$$

$$= -\cos\theta\sin^2\phi\vec{i} - \sin^2\theta\sin\phi\cos\phi\vec{k} - \cos^2\theta\cos\phi\sin\phi\vec{k} - \sin\theta\sin^2\phi\vec{j}$$

$$= -\cos\theta\sin^2\phi\vec{i} - \sin\theta\sin^2\phi\vec{j} - \sin\phi\cos\phi\vec{k}$$

We must check to see if $\vec{N}$ points inward or outward. Let $\theta = 0$ and $\phi = \frac{\pi}{2}$. Then $\vec{N} = -\vec{i}$, so $\vec{N}$ is inner. Hence we use

$$\vec{N} = \cos\theta\sin^2\phi\vec{i} + \sin\theta\sin^2\phi\vec{j} + \sin\phi\cos\phi\vec{k}$$

$$\operatorname{curl}\vec{F} \cdot \vec{N} = -\sin\theta\sin^2\phi + 3\sin\phi\cos\phi$$

$$\int\int_S \operatorname{curl}\vec{F} \cdot \vec{n}ds = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left(-\sin\theta\sin^2\phi + 3\sin\phi\cos\phi\right)d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta|_0^{2\pi}\sin^2\phi d\phi + 3(2\pi)\int_0^{\frac{\pi}{2}}\sin\phi\cos\phi$$

$$= 0 + 3\pi\sin^2\phi\,\Big|_0^{\frac{\pi}{2}} = 3\pi$$

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Problem 6

a) (**13 points**)

It can be shown the for the force field

$$\vec{F}(x,y,z) = e^x \cos y\vec{i} - e^x \sin y\vec{j} + 2\vec{k}$$

that

$$\operatorname{curl}\vec{F} = 0$$

Find the work done by $\vec{F}$ on an object moving along a curve C from $\left(0, \frac{\pi}{2}, 1\right)$ to $(1, \pi, 3)$.

Solution: Since $\operatorname{curl}\vec{F} = 0$ this means that there exists $f(x,y,z)$ such that

$$\nabla f = \vec{F}$$

Hence

$$\text{Work} = \int \vec{F} \cdot dr = f(1,\pi,3) - f\left(0, \frac{\pi}{2}, 1\right)$$

We have to find f

$$f_x = e^x \cos y \quad f_y = -e^x \sin y \quad f_z = 2$$

Holding y and z and integrating f_x with respect to x leads to

$$f = e^x \cos y + g(y,z)$$

So

$$f_y = -e^x \sin y + g_y = -e^x \sin y$$

Therefore $g_y = 0$ and $g = h(z)$.

$$f = e^x \cos y + h(z)$$

$$f_z = h'(z) = 2$$

and $h(z) = 2z + K,$ where K is a constant.

$$f = e^x \cos y + 2z + K$$

$$\text{Work} = \int \vec{F} \cdot dr = f(1,\pi,3) - f\left(0, \frac{\pi}{2}, 1\right)$$

$$= -e + 6 - (0 + 2) = 4 - e$$

6 b) (**12 points**)

Evaluate

$$\oint_C \left(\arctan x + y^2\right)dx + \left(e^y - x^2\right)dy$$

where C is the path enclosing the annular region shown below, positively oriented.

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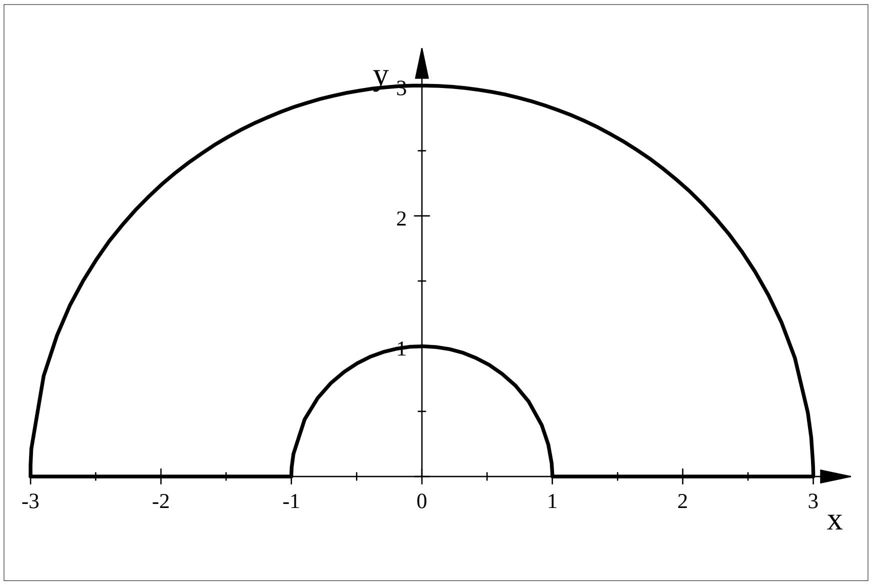

Solution: We use Green's Theorem, namely,

$$\oint P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Where R is the region enclosed by the closed curve C.

$$Q_x = -2x \text{ and } P_y = 2y$$

so

$$\oint_C \left(\arctan x + y^2\right) dx + \left(e^y - x^2\right) dy = -2 \iint_R (x+y) dA$$

$$= -2 \int_0^{\pi} \int_1^3 (r\cos\theta + r\sin\theta) r dr d\theta$$

$$= -2 \int_0^{\pi} (\cos\theta + \sin\theta) \frac{r^3}{3} \Big|_1^3 d\theta$$

$$= -\frac{52}{3} (\sin\theta - \cos\theta)|_0^{\pi} = -\frac{104}{3}$$

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Problem 7

a) **(13 points)**

Give and integral in polar coordinates for

$$\iint_R \left(x^2+y^2\right)^{-2} dA$$

into polar coordinates, where R is the part of the circle centered at $(1,0)$ of radius 1 to the right of the line $x = 1$ in the first quadrant. Be sure to sketch R. Do not evaluate this integral.

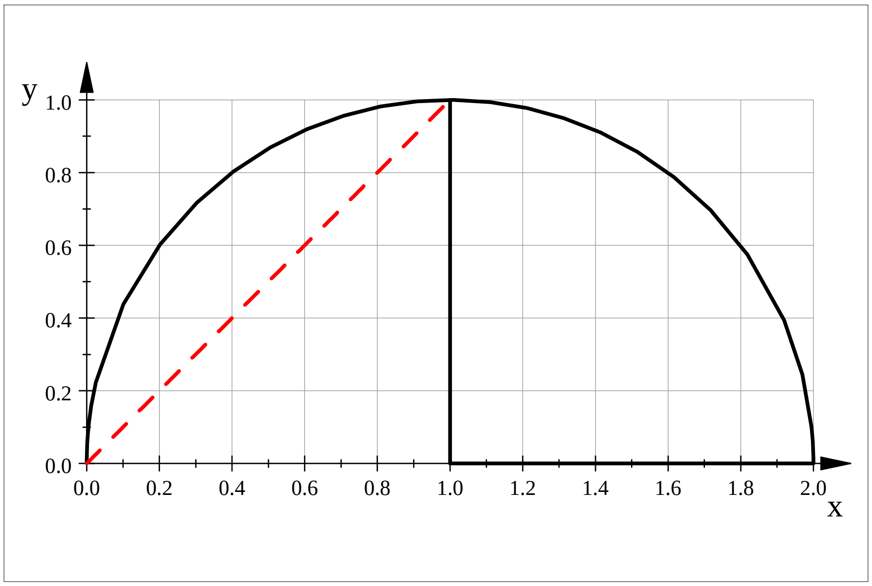
Solution: The equation of the circle is

$$(x-1)^2+y^2 = 1$$

or

$$x^2+y^2 = 2x$$

$\sqrt{2x-x^2}$



We must use polar coordinates to evaluate this integral. The equation of the circle in polar coordinates is $r = 2\cos\theta$. The equation of the line $x = 1$ is $r\cos\theta = 1$ or $r = \sec\theta$. r goes from the line to the circle, so $\sec\theta \le r \le 2\cos\theta$. The line $x = 1$ and the circle intersect at $(1,1)$. Thus θ goes from 0 to $\frac{\pi}{4}$, the angle that the line joining the origin and the point $(1,1)$, which is $y = x$ makes. Thus

$$\iint_R \left(x^2+y^2\right)^{-2} dA = \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} \left(r^2\right)^{-2} r\,dr\,d\theta = \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} r^{-3}\,dr\,d\theta$$

b) **(12 points)**

Evaluate

$$\iiint_E z\,dV$$

where E is the region within the cylinder $x^2+y^2 = 4$, where $0 \le z \le y$.

Solution: The condition $0 \le z \le y$ means that $y \ge 0$, so E projects onto the semicircle $y = \sqrt{1-x^2}$ in the first two quadrants of the $x,y-plane$.

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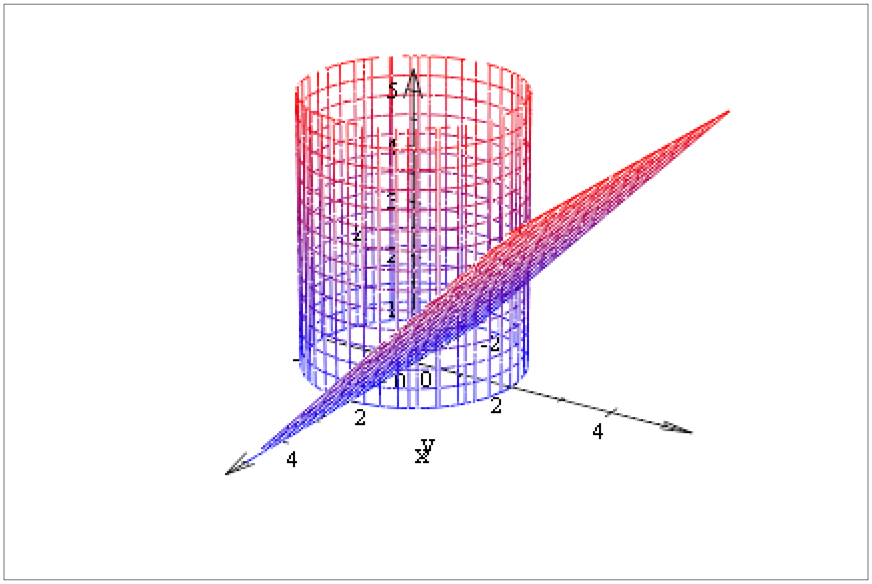
$z - y = 0$



The equation of the cylinder in cylindrical coordinates is $r = 2$, and the equation of the plane $z = y$ is $z = r\sin\theta$. Since $y \geq 0$ this means $0 \leq \theta \leq \pi$. The circle has radius 2, so $0 \leq r \leq 2$. Finally $0 \leq z \leq r\sin\theta$. Hence

$$\iiint_E z\,dV = \int_0^{\pi}\int_0^{2}\int_0^{r\sin\theta} zrdzdrd\theta$$

$$= \int_0^{\pi}\int_0^{2}\left(\frac{r\sin\theta}{2}\right)^2 rdrd\theta$$

$$= \frac{1}{2}\int_0^{\pi}\int_0^{2} r^3\sin^2\theta drd\theta$$

$$= \frac{1}{2}\frac{2^4}{4}\int_0^{\pi}\sin^2\theta d\theta$$

$$= 2\left(-\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\right)\Big|_0^{\pi} = \pi$$

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Problem 8

a) (**13 points**)

It can be shown that the Cauchy-Euler system

$$tx'(t) = Ax(t) \quad t > 0$$

where A is a constant matrix, has a nontrivial solutions of the form

$$x(t) = t^r u$$

if and only if r is an eigenvalue of A and u is a corresponding eigenvector. Use this information to solve the system

$$tx'(t) = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} x(t) \quad t > 0$$

Solution: We first find the eigenvectors of $\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$.

$$\left| \begin{bmatrix} -4-r & 2 \\ 2 & -1-r \end{bmatrix} \right| = (4+r)(1+r) - 4 = r^2 + 5r$$

so the eigenvalues are $r = 0, -5$. The system $(A - rI)x = 0$ is

$$(-4-r)x_1 + 2x_2 = 0$$

$$2x_1 + (-1-r)x_2 = 0$$

$r = 0$ yields $-4x_1 + 2x_2 = 0$ or $2x_1 = x_2$. This gives the eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. $r = -5$ leads to the equation $x_1 = -2x_2$ and hence the eigenvector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Thus

$$x(t) = c_1 t^0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 t^{-5} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

b) (**12 points**)

Rewrite the system of equations

$$x' = Ax + f$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \text{ and } f = \begin{bmatrix} 0 \\ 0 \\ \cos t \end{bmatrix}$$

as a single scalar equation.

Solution: We have

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$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos t \end{bmatrix}$$

Since

$$x_1(t) = y(t),\ \ x_2(t) = y'(t),\ x_3(t) = y''(t)$$

so

$$\begin{aligned} x_1'(t) &= y'(t) = x_2(t) \\ x_2'(t) &= y''(t) = x_3(t) \\ x_3'(t) &= y'''(t) = y' - y + \cos t = x_2 - x_1 + \cos t \end{aligned}$$

Thus

$$\frac{d^3y}{dt^3} - \frac{dy}{dt} + y = \cos t$$

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Table of Integrals

$$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$$
$$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$$
$$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$$
$$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$$
$$\int \left(\cos^2 x - \sin^2 x\right) dt = \frac{1}{2}\sin 2x + C$$