Ma 227 **Final Exam Solutions** **12/16/10**

Name: ________________________________

Lecture Section: ____ (**A**: **Prof**. **Levine**, **B**: **Prof**. **Brady**, **C**: **Prof**. **Strigul**)

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Directions: Answer all questions. The point value of each problem is indicated. If you need more work space, continue the problem you are doing on the **other side of the page it is on**.

There is a table of integrals at the end of the exam.

Score on Problem #1 ________

#2a ________

#2b ________

#3 ________

#4 ________

#5 ________

#6 ________

#7 ________

#8 ________

Total ________

I pledge my honor that I have abided by the Stevens Honor System.

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Problem 1

a) (**13 points**)

Find the value of c that makes it possible to solve the following system of equations and solve the system. Is your solution unique or an infinite set?

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= 5 \\ 3x_1 + 4x_2 + x_3 &= c \end{aligned}$$

Solution: We perform Gaussian elimination on the augmented matrix.

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 3 & -1 & 5 \\ 3 & 4 & 1 & c \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 1 & -5 & c-6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & c-7 \end{bmatrix}$$

The last row is equivalent to $0 = c - 7$. So there are solutions only for $c = 7$, and in this case, we have an infinite set of solutions. Setting $c = 7$ we continue the process to obtain the reduced row-echelon form fro which we can read off the solutions..

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & 1 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are pivots in columns 1 and 2, so the third variable is arbitrary.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - 7x_3 \\ 1 + 5x_3 \\ x_3 \end{bmatrix}$$

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b) (**12 points**)

Let $\vec{F}(x,y,z)$ and $\vec{G}(x,y,z)$ be vector fields with continuous partial derivatives. Show that

$$div\left(\vec{F}\times\vec{G}\right) = \vec{G}\bullet curl\vec{F} - \vec{F}\bullet curl\vec{G}$$

Solution: Let $\vec{F}(x,y,z) =< P,Q,R >$ and $\vec{G}(x,y,z) =< A,B,C >$.

$$\vec{F}\times\vec{G} = \begin{vmatrix} i & j & k \\ P & Q & R \\ A & B & C \end{vmatrix}$$

$$= (QC-RB)\vec{i} - (PC-RA)\vec{j} + (PB-QA)\vec{k}$$

$$div\left(\vec{F}\times\vec{G}\right) = Q_xC + QC_x - R_xB - RB_x - P_yC - PC_y + R_yA + RA_y + P_zB + PB_z - Q_zA - QA_z$$

$$=< A,B,C > \bullet < R_y - Q_z, P_z - R_x, Q_x - P_y > - < P,Q,R > \bullet < C_y - B_z, A_z - C_x, B_x - A_y >$$

$$= \vec{G}\bullet curl\vec{F} - \vec{F}\bullet curl\vec{G}$$

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Problem 2

a) (**10 points**) **Find the eigenvalues and eigenvectors of the matrix A**.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Solution. First we find the eigenvalues.

$$Det(A - rI) = \begin{vmatrix} 2-r & -1 & 0 \\ 2 & 1-r & 1 \\ 0 & 2 & 1-r \end{vmatrix}$$

$$= (2-r)\begin{vmatrix} 1-r & 1 \\ 2 & 1-r \end{vmatrix} + (1)\begin{vmatrix} 2 & 1 \\ 0 & 1-r \end{vmatrix}$$

$$= (2-r)\left[\left(1 - 2r + r^2\right) - 2\right] + 2(1-r)$$

$$= (2 - 4 + 2) + (-4 - 1 + 2 - 2)r + (2+2)r^2 - r^3$$

$$= -5r + 4r^2 - r^3 = -r\left(r^2 - 4r + 5\right)$$

Clearly one root is $r = 0$. Using the quadratic formula, the others are

$$r = \frac{4 \pm \sqrt{4^2 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2}$$

$$= 2 \pm i$$

The system of equations for the eigenvectors is

$$(2-r)u_1 - u_2 = 0$$

$$2u_1 + (1-r)u_2 + u_3 = 0$$

$$2u_2 + (1-r)u_3 = 0$$

For $r = 0$, we solve

$$(A - 0I)u = 0$$

Using elimination on the augmented matrix, we have

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -.5 & 0 & 0 \\ 0 & 1 & .5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & .25 & 0 \\ 0 & 1 & .5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The third component is arbitrary, so any multiple of

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$$u = \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$$

is an eigenvector.

Similarly, for $r = 2 + i$, we have the following. [The first step is an extra step of multiplying the first row by $2i$ to show how this goes.]

$$\begin{bmatrix} -i & -1 & 0 & 0 \\ 2 & -1-i & 1 & 0 \\ 0 & 2 & -1-i & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -2i & 0 & 0 \\ 2 & -1-i & 1 & 0 \\ 0 & 2 & -1-i & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -2i & 0 & 0 \\ 0 & -1+i & 1 & 0 \\ 0 & 2 & -1-i & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 2 & -2i & 0 & 0 \\ 0 & 2 & -1-i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 1-i & 0 \\ 0 & 2 & -1-i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Again, the third component is arbitrary and any multiple of

$$\begin{bmatrix} -1+i \\ 1+i \\ 2 \end{bmatrix}$$

is an eigenvector.

Finally, since the entries in the matrix are all real, both eigenvalues and eigenvectors come in complex conjugate pairs and for $r = 2 - i$, eigenvectors are multiples of

$$\begin{bmatrix} -1-i \\ 1-i \\ 2 \end{bmatrix}.$$

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b) (**15 points**)

The eigenvalues of the matrix $\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ are $2 + i$ and $2 - i$ and the corresponding eigenvectors are $\begin{bmatrix} -1-i \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1+i \\ 2 \end{bmatrix}$.

Find a [real] general solution to

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 25t \\ 0 \end{bmatrix}.$$

Solution: First we find a real general solution to $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. We expand one of the complex solutions and take the real and imaginary parts.

$$\begin{aligned}
e^{(2+i)t}\begin{bmatrix} -1-i \\ 2 \end{bmatrix} &= e^{2t}(\cos t + i\sin t)\begin{bmatrix} -1-i \\ 2 \end{bmatrix} \\
&= e^{2t}\begin{bmatrix} (-\cos t + \sin t) + i(-\sin t - \cos t) \\ (2\cos t) + i(2\sin t)) \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(-\cos t + \sin t) \\ 2e^{2t}\cos t \end{bmatrix} + i\begin{bmatrix} e^{2t}(-\sin t - \cos t) \\ 2e^{2t}\sin t \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{x}_h &= c_1\begin{bmatrix} e^{2t}(-\cos t + \sin t) \\ 2e^{2t}\cos t \end{bmatrix} + c_2\begin{bmatrix} e^{2t}(-\sin t - \cos t) \\ 2e^{2t}\sin t \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(-\cos t + \sin t) & e^{2t}(-\sin t - \cos t) \\ 2e^{2t}\cos t & 2e^{2t}\sin t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}
\end{aligned}$$

Next, we find a particular solution to the given non-homogeneous equation. Since the non-homogeneous term is a polynomial of degree one, the solution must be the same.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} at + b \\ ct + d \end{bmatrix}$$

Substitute into the system of d.e.s and find the coefficients.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 25t \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} at + b \\ ct + d \end{bmatrix} + \begin{bmatrix} 25t \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} (3a + c)t + (3b + d) \\ (-2a + c)t + (-2b + d) \end{bmatrix} + \begin{bmatrix} 25t \\ 0 \end{bmatrix}$$

We equate like terms.

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$$\begin{aligned} 0 &= 3a + c + 25 \\ 0 &= -2a + c \\ a &= 3b + d \\ c &= -2b + d \end{aligned}$$

Thus (from the first pair of equations) $a = -5,\ c = -10$ and then $b = 1$ and $d = -8$. Combining homogeneous and particular solutions, we have a general solution.

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= c_1 \begin{bmatrix} e^{2t}(-\cos t + \sin t) \\ 2e^{2t}\cos t \end{bmatrix} + c_2 \begin{bmatrix} e^{2t}(-\sin t - \cos t) \\ 2e^{2t}\sin t \end{bmatrix} + \begin{bmatrix} -5t + 1 \\ -10t - 8 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t}(-\cos t + \sin t) & e^{2t}(-\sin t - \cos t) \\ 2e^{2t}\cos t & 2e^{2t}\sin t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -5t + 1 \\ -10t - 8 \end{bmatrix} \end{aligned}$$

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Problem 3

a) (**25 points**)

Consider the triple integral

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \left(x^2+y^2+z^2\right) dz\,dy\,dx.$$

i. Describe and sketch the region of integration.

ii. Give an equivalent triple integral in rectangular coordinates in a different order of integration.

iii. Give an equivalent triple integral in cylindrical coordinates.

iv. Give an equivalent triple integral in spherical coordinates.

v. Use any of your equivalent triple integrals to evaluate the integral.

Solution: i. The region is the portion in the first octant of the ball of radius 1 centered at the origin.

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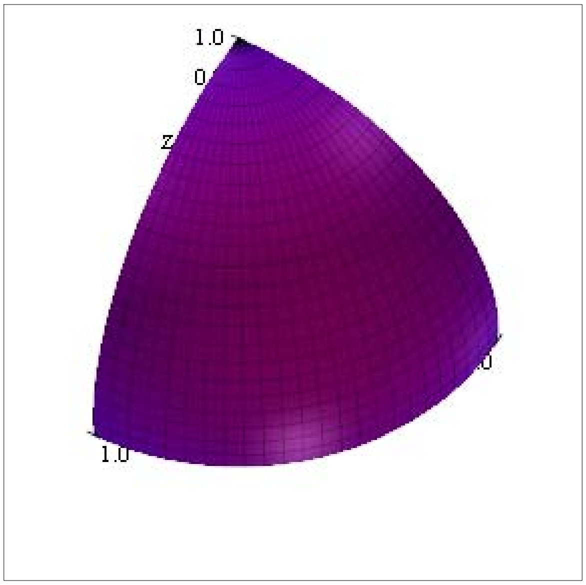

ii. There are five more possibilities, not too different. Two are

$$I = \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-y^2-z^2}} \left(x^2+y^2+z^2\right) dx\,dy\,dz$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} \left(x^2+y^2+z^2\right) dz\,dx\,dy$$

iii.

$$I = \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\sqrt{1-r^2}} \left(r^2+z^2\right) r\,dz\,dr\,d\theta$$

iv.

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$$I = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^4 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

v. Spherical coordinates seems easiest.

$$I = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left[\frac{\rho^5}{5} \right]_{\rho=0}^{\rho=1} \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{5} \int_0^{\frac{\pi}{2}} [-\cos\phi]_{\phi=0}^{\phi=\frac{\pi}{2}} d\theta$$

$$= \frac{1}{5} \theta \Big|_{\theta=0}^{\theta=\frac{\pi}{2}} = \frac{\pi}{10}$$

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Problem 4

a) (**10 points**)

Evaluate $\int_C (2x-y)dx + (-x-2y)dy$ where C is given by $x = \cos\theta$, $y = \sin^2\theta$, $0 \le \theta \le \frac{\pi}{2}$.

Solution: Let $P = 2x - y$, $Q = -x - 2y$. Since $P_y = Q_x = -1$, the integral is independent of path. The integral is easily evaluated using the straight line, C_1, from $(1,0)$ to $(0,1)$.

C_1 : $x = 1 - t$, $y = t$, $0 \le t \le 1$.

$$\begin{aligned}\int_C (2x-y)dx + (-x-2y)dy &= \int_{C_1} (2x-y)dx + (-x-2y)dy \\ &= \int_0^1 (2-2t-t)(-1)dt + (-1+t-2t)dt \\ &= \int_0^1 (-3+2t)dt = -3t + t^2 \Big|_{t=0}^{t=1} \\ &= -3 + 1 = -2\end{aligned}$$

Alternative solution: We can also use the given parametrization. $dx - -\sin\theta d\theta$, $dy = 2\sin\theta\cos\theta d\theta$

$$\begin{aligned}\int_C (2x-y)dx + (-x-2y)dy &= \int_0^{\frac{\pi}{2}} \Big[\left(2\cos\theta - \sin^2\theta\right)(-\sin\theta) + \left(-\cos\theta - 2\sin^2\theta\right)(2\sin\theta\cos\theta) \Big] d\theta \\ &= \int_0^{\frac{\pi}{2}} \left[-2\cos\theta\sin\theta + \sin^3\theta - 2\cos^2\theta\sin\theta - 4\sin^3\theta\cos\theta\right] d\theta \\ &= \left[\cos^2\theta - \frac{1}{3}\sin^2\theta\cos\theta - \frac{2}{3}\cos\theta + \frac{2}{3}\cos^3\theta - \sin^4\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}} \\ &= \left[0 - \frac{1}{3}\cdot 0 - \frac{2}{3}\cdot 0 + \frac{2}{3}\cdot 0 - 1\right] - \left[1 - \frac{1}{3}\cdot 0 - \frac{2}{3}\cdot 1 + \frac{2}{3}\cdot 1 - 0\right] \\ &= -2\end{aligned}$$

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b) (**15 points**)

Evaluate the surface integral $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS$ for $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ with S the triangle with vertices $(1,0,0), (0,1,0)$ and $(0,0,1)$ oriented upward.

Solution: The equation of the plane containing the triangle is $x + y + z = 1$. For S we use the parametrization

$$x = x$$
$$y = y$$
$$z = 1 - x - y$$
$$0 \le x \le 1,\ 0 \le y \le 1 - x$$

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + (1 - x - y)\mathbf{k}$$
$$\mathbf{r}_x = \mathbf{i} - \mathbf{k}$$
$$\mathbf{r}_y = \mathbf{j} - \mathbf{k}$$
$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

We observe that the normal vector has the correct (upward) direction and continue.

$$\begin{aligned}
\iint_S \vec{F} \cdot d\vec{S} &= \int_0^1 \int_0^{1-x} [(1 - x - y)\mathbf{i} + x\mathbf{j} + y\mathbf{k}] \cdot [\mathbf{i} + \mathbf{j} + \mathbf{k}]dydx \\
&= \int_0^1 \int_0^{1-x} (1 - x - y + x + y)dydx \\
&= \int_0^1 \int_0^{1-xx} dydx \\
&= \int_0^1 [y]_{y=0}^{y-1-x} dx \\
&= \int_0^1 [1 - x]dx \\
&= x - \frac{1}{2}x^2 \Big|_{x=0}^{x=1} \\
&= \frac{1}{2}
\end{aligned}$$

Problem 5

(25 points)

Verify the Divergence Theorem for the vector field $\vec{F} = y\vec{i} + yz\vec{j} + z^2\vec{k}$ where S is the surface of the solid E bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 5$.

Solution: The Divergence Theorem is

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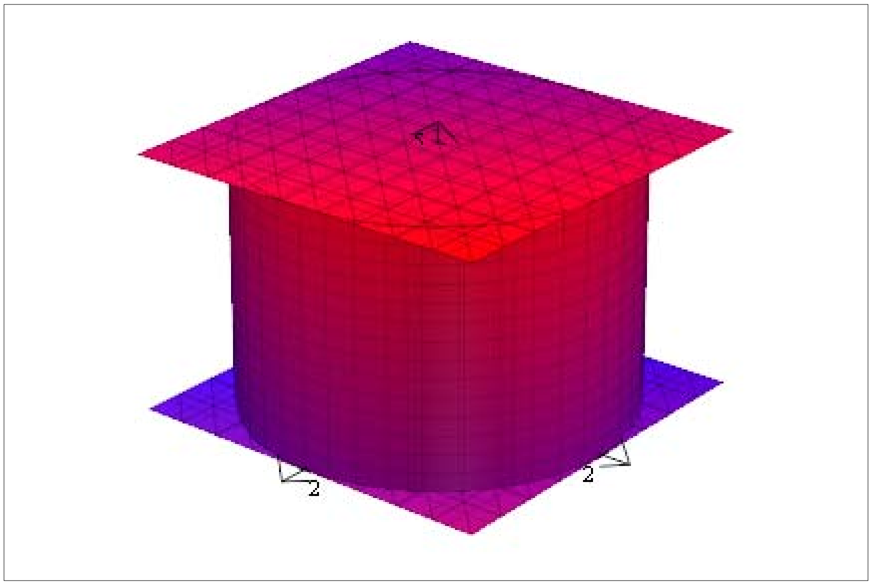
$$\iiint_E \nabla \cdot \vec{F}dV = \iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}dS$$

E is shown below.
$x^2 + y^2 = 4$



$S,$ the surface of $E,$ consists of three parts - the side of the cylinder, the bottom, and the top.
I. Side of the cylinder: We use cylindrical coordinates to parametrize the side of the cylinder. Let $x = 2\cos\theta, y = 2\sin\theta, z = z$ so that

$$\vec{r}(\theta,z) = 2\cos\theta\vec{i} + 2\sin\theta\vec{j} + z\vec{k}$$

$$\vec{r}_\theta = -2\sin\theta\vec{i} + 2\cos\theta\vec{j}$$

$$\vec{r}_z = \vec{k}$$

Thus

$$\vec{r}_\theta \times \vec{r}_z = \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{array} \right| \begin{array}{cc} \vec{i} & \vec{j} \\ -2\sin\theta & 2\cos\theta \\ 0 & 0 \end{array}$$

$$= 2\cos\theta\vec{i} + 2\sin\theta\vec{j}$$

For $\theta = 0$ $\vec{r}_\theta \times \vec{r}_z = 2\vec{i}$ which is outer. Hence

$$\iint_{side} \vec{F} \cdot \vec{n}dS = \int_0^5 \int_0^{2\pi} \left(2\sin\theta\vec{i} + 2z\sin\theta\vec{j} + z^2\vec{k}\right) \cdot \left(2\cos\theta\vec{i} + 2\sin\theta\vec{j}\right)d\theta dz$$

$$= \int_0^5 \int_0^{2\pi} \left(4\sin\theta\cos\theta + 4z\sin^2\theta\right)d\theta dz$$

$$= 0 + 4\int_0^5 \left[-\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\right]_0^{2\pi} dz = 4\pi\int_0^5 zdz = 4\pi\frac{25}{2} = 50\pi$$

II. Top of the cylinder: Here $z = 5,$ $\vec{n} = \vec{k}$ and $\vec{F} = y\vec{i} + 5y\vec{j} + 25\vec{k}$ so $\vec{F} \cdot \vec{n} = 25$ and

$$\iint_{top} \vec{F} \cdot \vec{n}dS = 25\iint_{top} dA = 25\left(\text{Area of the circle } x^2 + y^2 = 4\right) = 25(4\pi) = 100\pi$$

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III. Bottom of cylinder. Here $z = 0$, $\vec{n} = -\vec{k}$ and $\vec{F} = y\vec{i}$, so $\vec{F} \cdot \vec{n} = 0$ and

$$\iint_{bottom} \vec{F} \cdot \vec{n} dS = 0$$

Thus

$$\iint_{S} \vec{F} \cdot \vec{n} dS = \iint_{side} \vec{F} \cdot \vec{n} dS + \iint_{top} \vec{F} \cdot \vec{n} dS + \iint_{bottom} \vec{F} \cdot \vec{n} dS$$

$$= 50\pi + 100\pi + 0 = 150\pi$$

We now calculate $\iiint_{E} \nabla \cdot \vec{F} dV$.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}\left(z^2\right) = 3z$$

$$\iiint_{E} \nabla \cdot \vec{F} dV = 3 \iint_{x^2+y^2\le 4} \int_0^5 z dz dA = 3\left(\frac{25}{2}\right) \iint_{x^2+y^2\le 4} dA$$

$$= \frac{75}{2}\left(\text{Area of the circle } x^2 + y^2 = 4\right) = \frac{75}{2}(4\pi) = 150\pi$$

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Problem 6 (25 **points**)

Verify Green's Theorem for the line integral

$$\oint_C \left(xy^2dx + xdy\right)$$

where C is the unit circle centered at the origin oriented counterclockwise.

Solution: We have to show

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$$

C is the circle $x^2 + y^2 = 1$. C may be parametrized by $x = \cos t, y = \sin t, 0 \le t \le 2\pi$.

$$\begin{aligned}\oint_C \left(xy^2dx + xdy\right) &= \int_0^{2\pi}\left[\cos t\sin^2 t(-\sin t) + \cos t(\cos t)\right]dt \\ &= \int_0^{2\pi}\left[-\cos t\sin^3 t + \cos^2 t\right]dt \\ &= \left[-\frac{\sin^4 t}{4} + \frac{1}{2}\cos t\sin t + \frac{1}{2}t\right]_0^{2\pi} = \pi\end{aligned}$$

Also, here $P = xy^2$ and $Q = x,$ so $P_y = 2xy$ and $Q_x = 1.$ Hence

$$\begin{aligned}\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA &= \iint_{x^2+y^2\le 1} (1-2xy)dA \\ &= \int_0^{2\pi}\int_0^1\left(1 - 2r^2\cos\theta\sin\theta\right)rdrd\theta \\ &= \int_0^{2\pi}\left[\frac{r^2}{2} - 2\frac{r^3}{3}\cos\theta\sin\theta\right]_0^1 d\theta \\ &= \left[\frac{1}{2}\theta + 2\frac{1}{12}\cos^2\theta\right]_0^{2\pi} = \pi\end{aligned}$$

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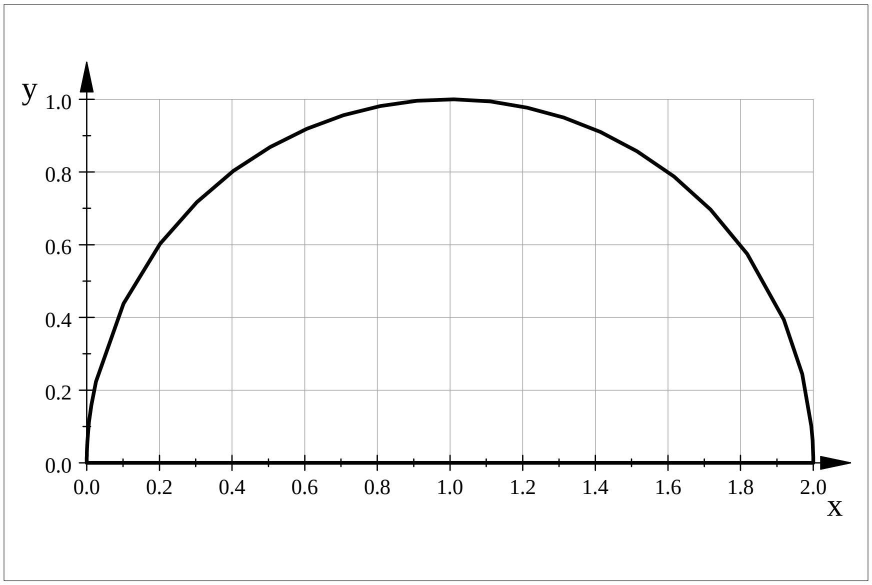
Problem 7

a) (**13 points**)
Calculate

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dydx$$

Solution: y goes from 0 to $y = \sqrt{2x - x^2}$ or the circle $x^2 - 2x + y^2 = 0$. In standard form the circle is $(x-1)^2 + y^2 = 1$. This is the circle of radius 1 and center at $(1,0)$. However, since the lower value of y is 0, the region is the top half of this circle and is shown below.
$\sqrt{2x - x^2}$



To evaluate the integral we switch to polar coordinates. The equation of the circle may be written as $x^2 + y^2 = 2x$ or $r = 2\cos\theta$. Thus, since the region is in the first quadrant

$$\begin{aligned}\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dydx &= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} (r) r dr d\theta \\ &= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3} \right]_0^{2\cos\theta} d\theta \\ &= \frac{8}{3} \int_0^{\frac{\pi}{2}} \cos^3\theta d\theta \\ &= \frac{8}{3} \left[\frac{1}{3}\cos^2\theta \sin\theta + \frac{2}{3}\sin\theta \right]_0^{\frac{\pi}{2}} = \frac{16}{9}\end{aligned}$$

b) (**12 points**)
Find the volume of solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. Sketch the solid.

Note: This is Example 4 on page 886 of Stewart.

Solution: The equation of the sphere is $x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$ so it has center at $\left(0, 0, \frac{1}{2}\right)$ and passes through the origin. The cone is a 45° cone The solid is shown below.

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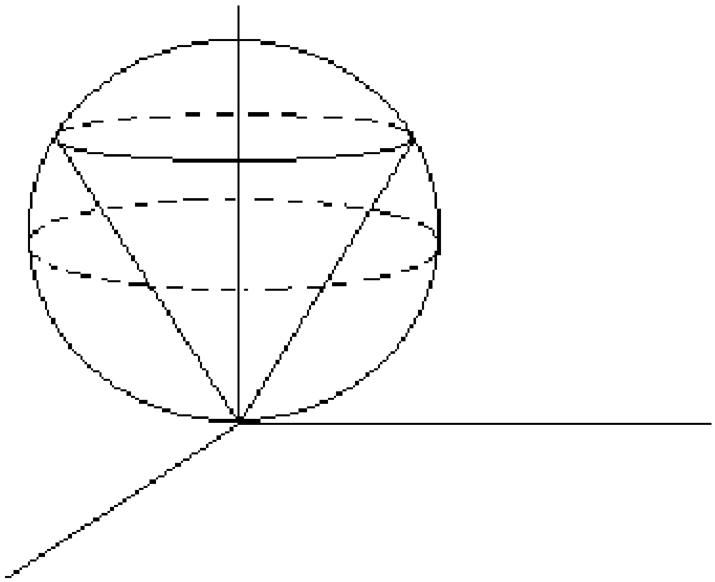

We use spherical coordinates. The equation of the sphere is

$$\rho^2 = \rho\cos\phi \text{ or } \rho = \cos\phi$$

The equation of the cone is

$$\rho\cos\phi = \sqrt{\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi} = \rho\sin\phi$$

Hence $\tan\phi = 1,$ so the equation of the cone in spherical coordinates is $\phi = \frac{\pi}{4}$.
The description of the solid E in spherical coordinates is

$$E = \left\{(\rho,\theta,\phi) | 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{4}, 0 \le \rho \le \cos\phi\right\}$$

Thus

$$V(E) = \iiint_E dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\phi\left[\frac{\rho^3}{3}\right]_0^{\cos\phi} d\phi$$

$$= \frac{2\pi}{3}\int_0^{\frac{\pi}{4}}\sin\phi\cos^3\phi\, d\phi = \frac{2\pi}{3}\left[-\frac{\cos^4\phi}{4}\right]_0^{\frac{\pi}{4}} = -\frac{2\pi}{12}\left[\frac{1}{4} - 1\right] = \frac{\pi}{8}$$

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Problem 8

a) (**13 points**)

Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Find e^{At}.

Solution:

$$e^{At} = I + At + A^2 \frac{t^2}{2!} + \cdots + A^n \frac{t^n}{n!} + \cdots = \sum_{n=0}^{\infty} A^n \frac{t^n}{n!}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}^3 = A^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

In general

$$A^n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (-1)^n & 0 & 0 \\ 0 & 0 & 2^n & 0 \\ 0 & 0 & 0 & (-2)^n \end{bmatrix}$$

Thus

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$$e^{At} = \sum_{n=0}^{\infty} A^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (-1)^n & 0 & 0 \\ 0 & 0 & 2^n & 0 \\ 0 & 0 & 0 & (-2)^n \end{bmatrix} \frac{t^n}{n!}$$

$$= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{t^n}{n!} & 0 & 0 & 0 \\ 0 & \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} & 0 & 0 \\ 0 & 0 & \sum_{n=0}^{\infty} 2^n \frac{t^n}{n!} & 0 \\ 0 & 0 & 0 & \sum_{n=0}^{\infty} (-2)^n \frac{t^n}{n!} \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{2t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{bmatrix}$$

b) (12 points)
Rewrite the equation

$$\frac{d^3y}{dt^3} - \frac{dy}{dt} + y = \sin t,\ y(0) = 1, y'(0) = 0, y''(0) = -4$$

as a system of differential equations in normal form with appropriate initial condition.
Solution: Let

$$x_1(t) = y(t),\ \ x_2(t) = y'(t),\ x_3(t) = y''(t)$$

so

$$x_1'(t) = y'(t) = x_2(t)$$
$$x_2'(t) = y''(t) = x_3(t)$$
$$x_3'(t) = y'''(t) = y' - y + \sin t = -x_1 + x_2 + \sin t$$

Thus the system is

$$x'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix}$$

Since

$$x_1(t) = y(t),\ \ x_2(t) = y'(t),\ x_3(t) = y''(t)$$

then the initial condition is

$$x(0) = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$$

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Table of Integrals

$$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$$

$$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$$

$$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$$

$$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$$

$$\int \left(\cos^2 x - \sin^2 x\right) dt = \frac{1}{2}\sin 2x + C$$