Ma 227 Final Exam Solutions 12/13/11

Name: ________________________________

Lecture Section: ____ (**A and B**: **Prof**. **Levine**, **C**: **Prof**. **Brady**)

Problem 1

a) (**10 points**)

Find the eigenvalues and eigenvectors of the matrix A.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -5 \\ 2 & 1 & -1 \end{bmatrix}$$

Solution. First we find the eigenvalues.

$$\det(A - rI) = \begin{vmatrix} 2-r & 0 & 0 \\ 0 & 3-r & -5 \\ 2 & 1 & -1-r \end{vmatrix}$$

$$= (2-r)\begin{vmatrix} 3-r & -5 \\ 1 & -1-r \end{vmatrix}$$

$$= (2-r)\left[\left(-3-2r+r^2\right)+5\right]$$

$$= (2-r)\left[\left(2-2r+r^2\right)\right]$$

Clearly one root is $r = 2$. Using the quadratic formula, the others are

$$r = \frac{2 \pm \sqrt{2^2-8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$$

$$= 1 \pm i$$

For $r = 2$, we solve

$$(A - 2I)u = 0$$

Using elimination on the augmented matrix, we have

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 2 & 1 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & -3 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus

$$u_1 + u_3 = 0$$

$$u_2 - 5u_3 = 0$$

The third component is arbitary, so any multiple of

$$u = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$$

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is an eigenvector.

Similarly, for $r = 1 + i$, we have the following. [The third step is an extra step of multipying the second row by $2 + i$ to show how this goes.]

$$\begin{bmatrix} 1-i & 0 & 0 & 0 \\ 0 & 2-i & -5 & 0 \\ 2 & 1 & -2-i & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2-i & -5 & 0 \\ 2 & 1 & -2-i & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2-i & -5 & 0 \\ 0 & 1 & -2-i & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -5(2+i) & 0 \\ 0 & 1 & -2-i & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -(2+i) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus

$$\begin{aligned} u_1 &= 0 \\ u_2 - (2+i)u_3 &= 0 \end{aligned}$$

Again, the third component is arbitrary and any multiple of

$$\begin{bmatrix} 0 \\ 2+i \\ 1 \end{bmatrix}$$

is an eigenvector.

Finally, since the entries in the matrix are all real, both eigenvalues and eigenvectors come in complex conjugate pairs and for $r = 1 - i,$ eigenvectors are multiples of

$$\begin{bmatrix} 0 \\ 2-i \\ 1 \end{bmatrix}.$$

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b) (**15 points**)

The eigenvalues of the matrix $\begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix}$ are $1 + 2i$ and $1 - 2i$ and the corresponding eigenvectors are $\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1+i \\ 1 \end{bmatrix}$.

Find a [real] general solution to

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 25t \\ 0 \end{bmatrix}.$$

Solution: First we find a real general solution to $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. A solution has the form $e^{rt}v$, where r is an eigenvalue and v is a corresponding eigenvector. We expand one of the complex solutions and take the real and imaginary parts.

$$\begin{aligned}
e^{(1+2i)t}\begin{bmatrix} -1-i \\ 1 \end{bmatrix} &= e^t(\cos 2t + i\sin 2t)\begin{bmatrix} -1-i \\ 1 \end{bmatrix} \\
&= e^t\begin{bmatrix} (-\cos 2t + \sin 2t) + i(-\sin 2t - \cos 2t) \\ (\cos 2t) + i(\sin 2t)) \end{bmatrix} \\
&= \begin{bmatrix} e^t(-\cos 2t + \sin 2t) \\ e^t\cos 2t \end{bmatrix} + i\begin{bmatrix} e^t(-\sin 2t - \cos 2t) \\ e^t\sin 2t \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{x}_h &= c_1\begin{bmatrix} e^t(-\cos 2t + \sin 2t) \\ e^t\cos 2t \end{bmatrix} + c_2\begin{bmatrix} e^t(-\sin 2t - \cos 2t) \\ e^t\sin 2t \end{bmatrix} \\
&= \begin{bmatrix} e^t(-\cos 2t + \sin 2t) & e^t(-\sin 2t - \cos 2t) \\ e^t\cos 2t & e^t\sin 2t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}
\end{aligned}$$

Next, we find a particular solution to the given non-homogeneous equation. Since the non-homogeneous term is a polynomial of degree one, the solution must be the same.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} at+b \\ ct+d \end{bmatrix}$$

Substitute into the system of d.e.s and find the coefficients.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 25t \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} at+b \\ ct+d \end{bmatrix} + \begin{bmatrix} 25t \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} (3a+4c)t + (3b+4d) \\ (-2a-c)t + (-2b-d) \end{bmatrix} + \begin{bmatrix} 25t \\ 0 \end{bmatrix}$$

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We equate like terms on each row. The coefficients of t .are listed in the first two rows below, then the constant terms.

$$\begin{aligned} 0 &= 3a + 4c + 25 \\ 0 &= -2a - c \\ a &= 3b + 4d \\ c &= -2b - d \end{aligned}$$

Thus (from the first pair of equations) $a = 5,\ c = -10$ and then $b = 7$ and $d = -4$. Combining homogeneous and particular solutions, we have a general solution.

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= c_1 \begin{bmatrix} e^t(-\cos 2t + \sin 2t) \\ e^t \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} e^t(-\sin 2t - \cos 2t) \\ e^t \sin 2t \end{bmatrix} + \begin{bmatrix} 5t + 7 \\ -10t - 4 \end{bmatrix} \\ &= \begin{bmatrix} e^t(-\cos 2t + \sin 2t) & e^t(-\sin 2t - \cos 2t) \\ e^t \cos 2t & e^t \sin 2t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 5t + 7 \\ -10t - 4 \end{bmatrix} \end{aligned}$$

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Problem 2

a) (**10 points**)

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + z^2\vec{k}$ and C one turn around the spiral $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$ from $(1,0,0)$ to $(1,0,2\pi)$.

Solution: In terms of the parameter, t, we have

$$\vec{F}(x,y,z) = \cos t\vec{i} + \sin t\vec{j} + t^2\vec{k}$$
$$d\vec{r} = \left(-\sin t\vec{i} + \cos t\vec{j} + \vec{k}\right)dt$$

So

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(\cos t\vec{i} + \sin t\vec{j} + t^2\vec{k}\right) \cdot \left(-\sin t\vec{i} + \cos t\vec{j} + \vec{k}\right)dt$$
$$= \int_0^{2\pi} \left(-\sin t \cos t + \sin t \cos t + t^2\right)dt$$
$$= \int_0^{2\pi} t^2 dt = \frac{1}{3}t^3 \Big|_0^{2\pi} = \frac{8}{3}\pi^3$$

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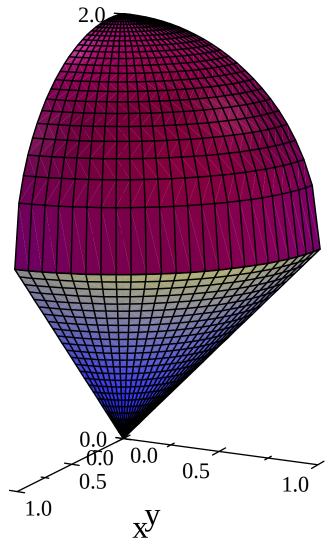
b) (**15 points**)
Consider the triple integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1+\sqrt{1-x^2-y^2}} \left(x^2+y^2+z^2\right) dzdydx.$$

i. Describe and sketch the region of integration.
Solution: The region of integration is in the first octant, bounded below by the cone $z^2 = x^2+y^2$ and above by the upper portion of the sphere of radius 1 centered at $(0,0,1)$ which has the equation

$$x^2+y^2+(z-1)^2 = 1.$$

(r, θ, r)



ii. Give an equivalent triple integral in cylindrical coordinates.
Solution: The cone is

$$z = r.$$

The sphere is

$$r^2 + (z-1)^2 = 1$$

The integral is

$$\int_0^{\frac{\pi}{2}} \int_0^1 \int_r^{1-\sqrt{1-r^2}} \left(r^2+z^2\right) r dr d\theta$$

iii. Give an equivalent triple integral in spherical coordinates.
Solution: In spherical coordinates, the equation of the cone is $\phi = \frac{\pi}{4.}$ The sphere requires a little algebra. We have

$$x^2+y^2+z^2-2z+1 = 1$$
$$\rho^2 - 2\rho\cos\phi = 0$$
$$\rho = 2\cos\phi$$

The integral is

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{2\cos\phi} \left(\rho^2\right) \rho^2 \sin\phi \, d\rho d\phi d\theta.$$

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Problem 3

(**25 points**)
Evaluate the surface integral

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$$

where S is the surface of the portion of the cone $z^2 = x^2 + y^2$ in the first octant and below the plane $z = 4$ with downward normal and

$$\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k}.$$

Solution: We need to parametrize the surface and then set up the integral, in terms of the parameters that we choose, in the form

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot \left(\vec{r}_u \times \vec{r}_v\right) dA_{uv}.$$

I chose to use cylindrical coordinates, since the equation of the cone is $z = r$.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = r$$

$$0 \le r \le 4 \quad 0 \le \theta \le \frac{\pi}{2}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = r\cos\theta\vec{i} + r\sin\theta\vec{j} + r\vec{k}$$

The computations are

$$\vec{r}_r = \cos\theta\vec{i} + \sin\theta\vec{j} + \vec{k}$$

$$\vec{r}_\theta = -r\sin\theta\vec{i} + r\cos\theta\vec{j}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= -r\cos\theta\,\vec{i} - r\sin\theta\,\vec{j} + r\vec{k}$$

We observe that we have a normal vector pointing upward from the positive z component and use the negative, i.e $\vec{r}_v \times \vec{r}_u$.

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$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{\frac{\pi}{2}} \int_0^4 \left(r^2 \sin\theta \vec{i} + r^2 \cos\theta \vec{j} + r^2 \cos\theta \sin\theta \vec{k} \right) \cdot \left(r\cos\theta \vec{i} + r\sin\theta \vec{j} - r\vec{k} \right) dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^4 \left(r^3 \sin\theta\cos\theta + r^3 \cos\theta \sin\theta - r^3 \sin\theta \cos\theta \right) dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^4 r^3 \sin\theta \cos\theta dr d\theta = \frac{4^4}{4} \int_0^{\frac{\pi}{2}} \sin\theta \cos\theta d\theta$$

$$= 64 \frac{\sin^2\theta}{2} \Big|_0^{\pi/2} = 32.$$

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Problem 4

a) (**10 points**)

Show that, if all partial derivatives of $f(x,y,z)$ are continuous,

$$\text{curl}\big(\text{grad } f(x,y,z)\big) = \vec{0}.$$

Solution:

$$grad\, f(x,y,z) = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$$

$$curl\big(grad\, f(x,y,z)\big) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$= (f_{zy} - f_{zy})\vec{i} - (f_{zx} - f_{xz})\vec{j} + (f_{yx} = f_{xy})\vec{k}$$

$$= \vec{0}$$

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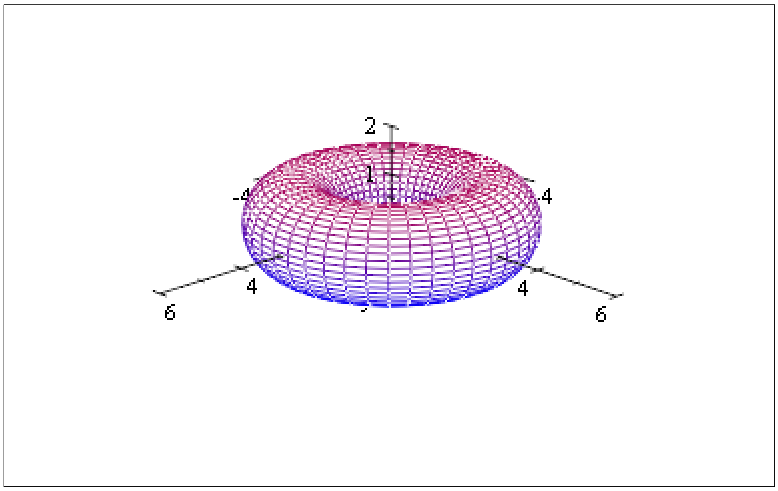
b) (**15 points**)

The figure shows the torus obtained by rotating about the *z*-axis the circle in the xz-plane with center $(2,0,0)$ and radius 1.
Parametric equations for the torus are

$$\begin{aligned} x &= 2\cos\theta + \cos\alpha\cos\theta \\ y &= 2\sin\theta + \cos\alpha\sin\theta \\ z &= \sin\alpha \\ 0 &\le \alpha \le 2\pi, \quad 0 \le \theta \le 2\pi. \end{aligned}$$

θ is the usual polar angle around the z axis and α is the angle around the circle in the $x-z$ plane. Find the surface area of the torus.



Solution: We have that surface area, for a surface given parametrically, is given by

$$\iint_S dS = \iint_R \left|\vec{r}_u \times \vec{r}_v\right| dA_{uv}.$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = (2\cos\theta + \cos\alpha\cos\theta)\vec{i} + (2\sin\theta + \cos\alpha\sin\theta)\vec{j} + \sin\alpha\vec{k}$$

$$\vec{r} = (2+\cos\alpha)\cos\theta\vec{i} + (2+\cos\alpha)\sin\theta\vec{j} + \sin\alpha\vec{k}$$

The parametrization is given, so we proceed with the computations.

$$\begin{aligned} \vec{r}_\theta &= (2+\cos\alpha)(-\sin\theta)\vec{i} + (2+\cos\alpha)\cos\theta\vec{j} \\ \vec{r}_\alpha &= -\sin\alpha\cos\theta\vec{i} - \sin\alpha\sin\theta\vec{j} + \cos\alpha\vec{k} \end{aligned}$$

$$\vec{r}_\theta \times \vec{r}_\alpha = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (2+\cos\alpha)(-\sin\theta) & (2+\cos\alpha)\cos\theta & 0 \\ -\sin\alpha\cos\theta & -\sin\alpha\sin\theta & \cos\alpha \end{vmatrix}$$

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$$\vec{r}_\theta \times \vec{r}_\alpha = \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ (2+\cos\alpha)(-\sin\theta) & (2+\cos\alpha)\cos\theta & 0 \\ -\sin\alpha\cos\theta & -\sin\alpha\sin\theta & \cos\alpha \end{array} \right| \begin{array}{cc} \vec{i} & \vec{j} \\ (2+\cos\alpha)(-\sin\theta) & (2+\cos\alpha)\cos\theta \\ -\sin\alpha\cos\theta & -\sin\alpha\sin\theta \end{array}$$

$$= (2+\cos\alpha)\cos\theta\cos\alpha\vec{i} + (2+\cos\alpha)\sin\alpha\sin^2\theta\vec{k} + (2+\cos\alpha)\sin\alpha\cos^2\theta\vec{k} + (2+\cos\alpha)\cos\alpha\sin\theta\vec{j}$$

$$= (2+\cos\alpha)\left[\cos\theta\cos\alpha\vec{i} + \sin\alpha\left(\sin^2\theta + \cos^2\theta\right)\vec{k} + \cos\alpha\sin\theta\vec{j}\right]$$

$$= (2+\cos\alpha)\left[\cos\theta\cos\alpha\vec{i} + \cos\alpha\sin\theta\vec{j} + \sin\alpha\vec{k}\right]$$

Thus

$$\begin{aligned} \left|\vec{r}_\theta \times \vec{r}_\alpha\right| &= \sqrt{(2+\cos\alpha)^2\left(\cos^2\theta\cos^2\alpha + \cos^2\alpha\sin^2\theta + \sin^2\alpha\right)} \\ &= (2+\cos\alpha)\sqrt{\cos^2\alpha\left(\sin^2\theta + \cos^2\theta\right) + \sin^2\alpha} \\ &= 2+\cos\alpha \end{aligned}$$

$$\begin{aligned} \iint_S dS &= \int_0^{2\pi}\int_0^{2\pi}(2+\cos\alpha)\,d\alpha\, d\theta = \int_0^{2\pi}\left(2\alpha + \sin\alpha\big|_{\alpha=0}^{\alpha=2\pi}\right)d\theta \\ &= 4\pi\int_0^{2\pi} d\theta = 8\pi^2. \end{aligned}$$

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Problem 5

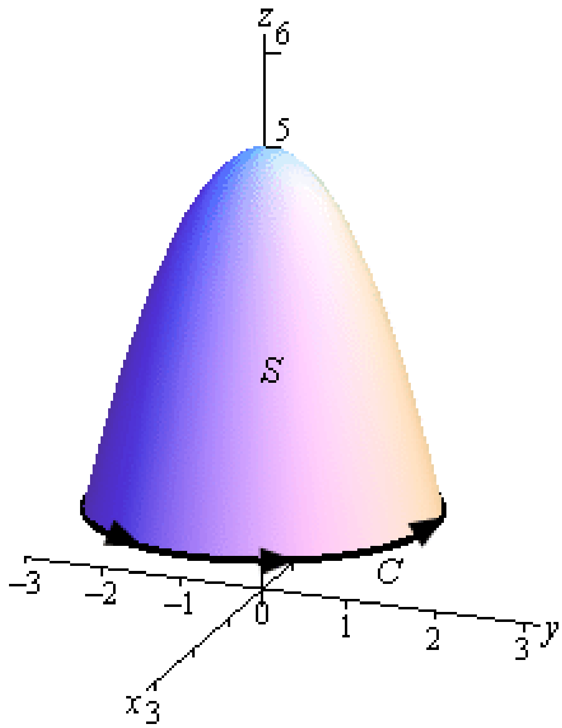
a) (**13 points**)

Use Stokes' Theorem to evaluate

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

where $\vec{F} = z^2\vec{i} - 3xy\vec{j} + x^3y^3\vec{k}$ and S is the part of $z = 5 - x^2 - y^2$ above the plane $z = 1$. Assume S is oriented upwards.

Solution: A sketch of the surface is shown below.



Stokes' Theorem is

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \oint_{\partial S} \vec{F}\cdot d\vec{r}$$

where ∂S is the boundary of the surface S.

The boundary of S is the curve where S intersects the plane $z = 1$. Thus we have $1 = 5 - x^2 - y^2$ or

$$\partial S:\ x^2 + y^2 = 4,\ \ z = 1$$

We parametrize ∂S by $x = 2\cos t, y = 2\sin t, 0 \le t \le 2\pi,\ z = 1$ so

$$\vec{r}(t) = 2\cos t\vec{i} + 2\sin t\vec{j} + \vec{k} \quad 0 \le t \le 2\pi$$

and

$$\vec{r}\,'(t) = -2\sin t\vec{i} + 2\cos t\vec{j}$$

Since $\vec{F} = z^2\vec{i} - 3xy\vec{j} + x^3y^3\vec{k},$ then

$$\vec{F}(t) = (1)^2\vec{i} - 3(2\cos t)(2\sin t)\vec{j} + (2\cos t)^3(2\sin t)^3\vec{k}$$

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Therefore

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(t) \cdot \vec{r}'(t)dt$$

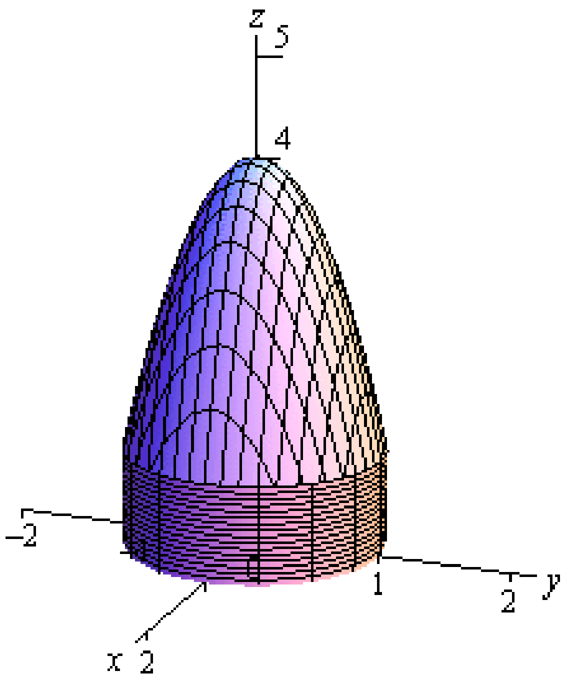
$$= \int_0^{2\pi} \left(-2\sin t - 24\sin t\cos^2 t\right)dt$$

$$= \left(2\cos t + 8\cos^3 t\right)\Big|_0^{2\pi} = 0$$

b) (12 points)
Use the divergence theorem to evaluate

$$\iint_S \vec{F} \cdot d\vec{S}$$

where $\vec{F} = xy\vec{i} - \frac{1}{2}y^2\vec{j} + z\vec{k}$ and the surface S consists of the three surfaces, $z = 4 - 3x^2 - 3y^2$, $1 \le z \le 4$, on the top, $x^2 + y^2 = 1$, $0 \le z \le 1$ on the sides, and $z = 0$ on the bottom.
Solution: A sketch of the surface is shown below.



The divergence theorem is

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}dV$$

where E is the volume enclosed by S.

We use cylindrical coordinates so that the limits for the variables are

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$$0 \le z \le 4 - 3r^2$$
$$0 \le r \le 1$$
$$0 \le \theta \le 2\pi$$

$$\text{div}\vec{F} = \nabla \cdot \vec{F} = y - y + 1$$

Hence

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\vec{F}dV$$
$$= \int_0^{2\pi}\int_0^1\int_0^{4-3r^2} rdzdrd\theta$$
$$= \int_0^{2\pi}\int_0^1 \left(4r - 3r^3\right)drd\theta$$
$$= \int_0^{2\pi}\left(2r^2 - \frac{3}{4}r^4\right)\Big|_0^1 d\theta = \frac{5}{4}\int_0^{2\pi} d\theta = \frac{5\pi}{2}$$

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Problem 6 (25 points)

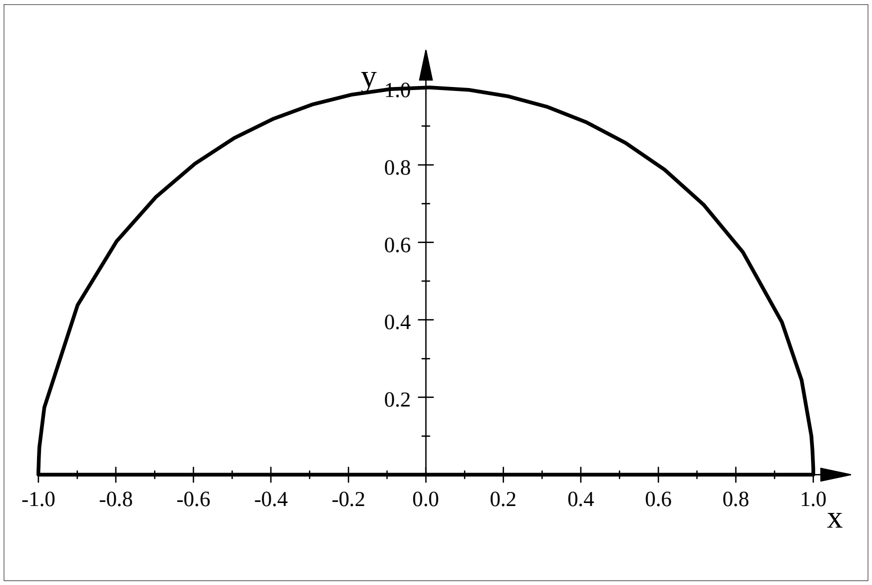
Verify Green's Theorem for the line integral

$$\oint_C \left(y^2dx + 3xydy\right)$$

where C is the closed curve consisting of the upper half of the unit circle centered at the origin oriented counterclockwise followed by the line joining $(-1,0)$ to $(1,0)$.

Solution: The closed curve C is shown below.

$\sqrt{1-x^2}$



Green's Theorem says

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$$

where R is the region enclosed by C.

$P = y^2$ and $Q = 3xy$ so

$$\begin{aligned}\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA &= \iint_R (3y-2y)dA \\ &= \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} 2ydydx \\ &= \int_{-1}^{1} \frac{y^2}{2}\Bigg|_0^{\sqrt{1-x^2}} dx \\ &= \frac{1}{2}\int_{-1}^{1}\left(1-x^2\right)dx = \frac{1}{2}\left(x - \frac{x^3}{3}\right)\Bigg|_{-1}^{1} = \frac{2}{3}\end{aligned}$$

We now compute $\oint_C \left(y^2dx + 3xydy\right)$ directly. Note the C consists of two parts, $C_1,$ the semicircle of unit radius from $(1,0)$ to $(-1,0),$ and C_2 the part of the x –axis from $(-1,0)$ to $(1,0)$.

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$$C_1 : x = \cos t, y = \sin t, \quad 0 \le t \le \pi$$
$$C_2 : x = t, y = 0 \quad -1 \le t \le 1$$

Thus

$$\oint_C \left(y^2 dx + 3xydy\right) = \int_0^\pi \left(\sin^2 t(-\sin t) + 3\cos t \sin t(\cos t)dt\right) + \int_{-1}^1 0dt$$
$$= \int_0^\pi \left(-\sin^3 t + 3\cos^2 t \sin t\right)dt = \int_0^\pi \left(-\sin t\left(1 - \cos^2 t\right) + 3\cos^2 t \sin t\right)dt$$
$$= \int_0^\pi \left(-\sin t + 4\cos^2 t \sin t\right)dt = \left(\cos t - \frac{4}{3}\cos^3 t\right)\Big|_0^\pi = -1 + \frac{4}{3} - \left(1 - \frac{4}{3}\right) = -2 + \frac{8}{3} = \frac{2}{3}$$

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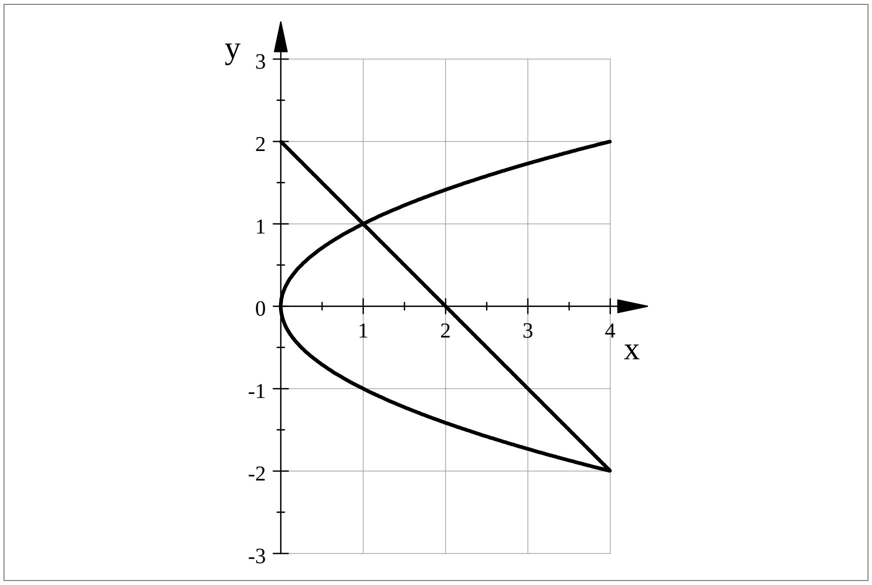
Problem 7

a) (**13 points**)
The integral

$$\int_{-2}^{1}\int_{y^2}^{2-y} dxdy$$

gives the area of a region R in the x,y –plane. Sketch R and then give another expression for the area of R with the order of integration reversed. Do *not* evaluate this expression.
Solution: This problem was on the Ma 227 final exam given in 04S. The curves that bound R are the parabola $x = y^2$ and the line $x = 2 - y$ or $y = 2 - x.$ Thus
$x = y^2$



The curves intersect when $y^2 = 2 - y$ or when $y^2 + y - 2 = (y+2)(y-1) = 0.$ Hence at $(1,1)$ and $(4,-2).$ Reversing the order of integration requires two integrals. They are

$$\int_{0}^{1}\int_{-\sqrt{x}}^{\sqrt{x}} dydx + \int_{1}^{4}\int_{-\sqrt{x}}^{2-x} dydx$$

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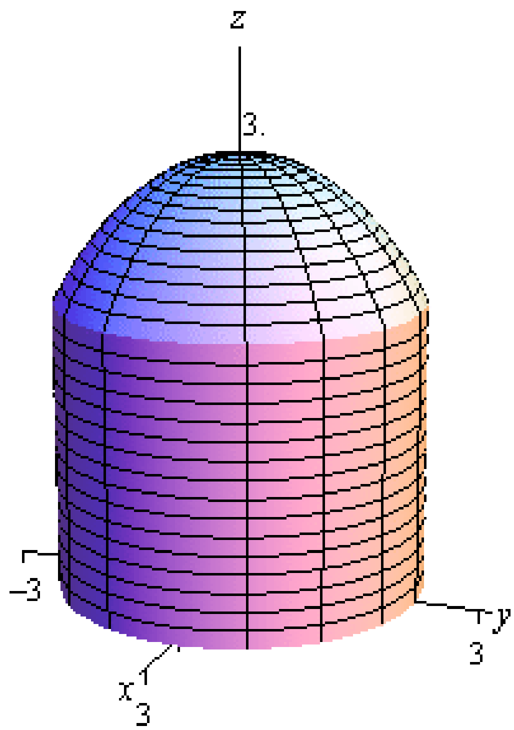
b) (**12 points**)

Find the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 5$. Sketch the solid.

Solution: The region is shown below.



We will use cylindrical coordinates.

$$V = \iiint_E dV = \iiint_E r\,dz\,dr\,d\theta$$

z goes from the x,y –plane to the sphere. In cylindrical coordinates the sphere is $z = \sqrt{9 - x^2 - y^2} = \sqrt{9 - r^2}$. Thus $0 \le z \le \sqrt{9 - r^2}$. The region on integration in the x,y –plane is the area $x^2 + y^2 \le 5$. Therefore $0 \le r \le \sqrt{5}$ and $0 \le \theta \le 2\pi$. Therefore

$$\begin{aligned}
V &= \iiint_E dV = \iiint_E r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{5}}\int_0^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{5}} r\sqrt{9-r^2}\,dr\,d\theta \\
&= \int_0^{2\pi} -\frac{(9-r^2)^{\frac{3}{2}}}{3}\Bigg|_0^{\sqrt{5}} d\theta = -\frac{1}{3}\int_0^{2\pi}\left(4^{\frac{3}{2}} - 9^{\frac{3}{2}}\right)d\theta \\
&= -\frac{2\pi}{3}(8 - 27) = \frac{38\pi}{3}
\end{aligned}$$

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Problem 8

a) (**8 points**)

Show that the characteristic polynomial for the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix}$$

is

$$p(r) = -(r-1)^3$$

Solution:The characteristic polynomial for A is

$$p(r) = \det\begin{bmatrix} 2-r & 1 & 1 \\ 1 & 2-r & 1 \\ -2 & -2 & -1-r \end{bmatrix}$$

$$= \left|\begin{bmatrix} 2-r & 1 & 1 \\ 1 & 2-r & 1 \\ -2 & -2 & -1-r \end{bmatrix}\right| \begin{matrix} 2-r & 1 \\ 1 & 2-r \\ -2 & -2 \end{matrix}$$

$$= -(2-r)^2(1+r) - 2 - 2 + 2(2-r) + 2(2-r) + 1 + r$$

$$= -\left(4 - 4r + r^2\right)(1+r) - 3 + 8 - 4r + r$$

$$= -4 + 4r - r^2 - 4r + 4r^2 - r^3 + 5 - 3r$$

$$= -r^3 + 3r^2 - 3r + 1 = -(r-1)^3$$

b) (**7 points**)

Show that

$$p(A) = -(A-I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where A is the matrix in part a)

Solution:

$$A - I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}$$

Thus

$$p(A) = --(A-I)^3 = -\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}^3$$

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$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ so } -\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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c) (**10 points**)
Calculate e^{At}.
Solution:

$$e^{At} = e^{t}e^{(A-I)t} = e^{t}\left[I + (A-I)t + (A-I)^{2}\frac{t^{2}}{2!} \right]$$

$$= e^{t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + te^{t}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} e^{t}+te^{t} & te^{t} & te^{t} \\ te^{t} & e^{t}+te^{t} & te^{t} \\ -2te^{t} & -2te^{t} & e^{t}-2e^{t} \end{bmatrix}$$

Name:________________________________

Table of Integrals

$$\int \sin^2 x\,dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$$

$$\int \cos^2 x\,dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$$

$$\int \sin^3 x\,dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$$

$$\int \cos^3 x\,dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$$

$$\int \left(\cos^2 x - \sin^2 x\right) dt = \frac{1}{2}\sin 2x + C$$