Ma 227 Final Exam Solutions 12/17/13

Name: ________________________________

Lecture Section: ____ (**A and B**: **Prof**. **Levine**, **C**: **Prof**. **Brady**)

Problem 1

a) (**12 points**)

Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{bmatrix}.$$

a) (**13 points**)
Find all eigenvalues and eigenvectors of the matrix A.

Solution: $\left\| \begin{bmatrix} 1-r & -1 & 0 \\ 1 & 2-r & 1 \\ -2 & 1 & -1-r \end{bmatrix} \right\| =$

$-r^3 + 2r^2 + r - 2 = -r^2(r-2) + r - 2 = (r-2)\left(-r^2+1\right)$, so $r = 2, 1, -1$

For $r = 2$, we obtain

$$\begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 1 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, $u_1 + u_3 = 0$ and $u_2 - u_3 = 0$. Choose $u_3 = 1$ and the other values follow.
Similarly, for $r = 1$, we obtain

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -2 & 1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and for $r = -1$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{7}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{7} & \\ 0 & 1 & \frac{2}{7} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The summary is (choosing the free variable - always the third here - to be one) given below. Any multiple of the eigenvectors may be used.

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{bmatrix}, \text{ eigenvectors: } \left\{ \begin{bmatrix} -\frac{1}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix} \right\} \leftrightarrow -1, \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \leftrightarrow 1, \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} \leftrightarrow 2$$

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b) (**13 points**)
Find a general solution for the system

$$\mathbf{x}'(t) = Ax(t), \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

where A is the matrix above.
Solution:

$$x(t) = c_1 \begin{bmatrix} -\frac{1}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{2t}$$

$$= \begin{bmatrix} -\frac{1}{7}e^{-t} & -1e^{t} & -1e^{2t} \\ -\frac{2}{7}e^{-t} & 0 & 1e^{2t} \\ 1e^{-t} & 1e^{t} & 1e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} -\frac{1}{7}c_1 - c_2 - c_3 \\ -\frac{2}{7}c_1 + c_3 \\ c_1 + c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Thus we have three equations for c_1, c_2, c_3

$$-\frac{1}{7}c_1 - c_2 - c_3 = 1$$

$$-\frac{2}{7}c_1 + c_3 = 2$$

$$c_1 + c_2 + c_3 = 3$$

, Solution is: $\left[c_1 = \frac{14}{3}, c_2 = -5, c_3 = \frac{10}{3} \right]$

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Problem 2

a) (**12 points**)

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

Solution: We solve $\det(A - rI) = 0.$

$$\begin{aligned} \det(A - rI) &= \begin{vmatrix} 2-r & -1 \\ 1 & 2-r \end{vmatrix} \\ &= (2-r)^2 + 1 \\ (2-r)^2 &= -1 \\ 2 - r &= \pm i \\ r &= 2 \pm i \end{aligned}$$

So, the eigenvalues are a complex conjugate pair. We find the eigenvector for one and take the complex conjugate to get the other. For $r = 2 + i,$ we solve

$$(A - rI)u = 0$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus we have the equations

$$\begin{aligned} -iu_1 - u_2 &= 0 \\ u_1 - iu_2 &= 0 \end{aligned}$$

The second row is redundant, so $-iu_1 - u_2 = 0$ or $u_2 = -i \cdot u_1.$ Hence any multiple of $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector for $r = 2 + i.$ Then an eigenvector corresponding to $r = 2 - i$ is $\begin{bmatrix} 1 \\ i \end{bmatrix}.$

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b) (**13 points**)
Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 3 & 4 \\ 1 & -2 & -1 \end{bmatrix}.$$

Eigenvectors and corresponding eigenvalues of A are

$$\left\{\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right\} \leftrightarrow 2, \left\{\begin{bmatrix} 0 \\ -1+i \\ 1 \end{bmatrix}\right\} \leftrightarrow 1-2i, \left\{\begin{bmatrix} 0 \\ -1-i \\ 1 \end{bmatrix}\right\} \leftrightarrow 1+2i.$$

Find all real solutions to the system of differential equations

$$\mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 4e^t \\ -6e^t \\ 8e^t \end{bmatrix}$$

Solution: The solution has the form

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t),$$

where $\mathbf{x}_h(t)$ is a general solution to the homogeneous system, $\mathbf{x}' = A\mathbf{x}$, and $\mathbf{x}_p(t)$ is one (particular) solution to the geven system. The homogeneous solution comes from the eigenvalues and eigenvectors, but we must separate the real and imaginary parts to obtain two L.I. solutions.We use the eigenvalue $1+2i$ and obtain

$$\begin{bmatrix} 0 \\ -1-i \\ 1 \end{bmatrix} e^{(1+2i)t} = \begin{bmatrix} 0 \\ (-1-i)e^t(\cos 2t + i\sin 2t)) \\ e^t(\cos 2t + i\sin 2t)) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ e^t(-\cos 2t + \sin 2t) + ie^t(-\cos 2t - \sin 2t) \\ e^t\cos 2t + ie^t\sin 2t \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ e^t(-\cos 2t + \sin 2t) \\ e^t\cos 2t \end{bmatrix} + i\begin{bmatrix} 0 \\ e^t(-\cos 2t - \sin 2t) \\ e^t\sin 2t \end{bmatrix}.$$

Now, for the particular solution, we seek a solution of the form

$$\mathbf{x}_p(t) = \mathbf{a}e^t = \begin{bmatrix} a_1e^t \\ a_2e^t \\ a_3e^t \end{bmatrix}.$$

Substituting this into the system will lead to a set of algebraic equations. The details are straight forward.

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$$\mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 4e^t \\ -6e^t \\ 8e^t \end{bmatrix}$$

$$\begin{bmatrix} a_1 e^t \\ a_2 e^t \\ a_3 e^t \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 3 & 4 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} a_1 e^t \\ a_2 e^t \\ a_3 e^t \end{bmatrix} + \begin{bmatrix} 4e^t \\ -6e^t \\ 8e^t \end{bmatrix}$$

Since e^t is non-zero, we divide it out and obtain

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2a_1 + 4 \\ 2a_1 + 3a_2 + 4a_3 - 6 \\ a_1 - 2a_2 - a_3 + 8 \end{bmatrix}$$

The elimination process is shown after putting the variables on one side and the constants on the other .

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 2 & 2 & 4 & 6 \\ 1 & -2 & -2 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 2 & 4 & 14 \\ 0 & -2 & -2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 2 & 10 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Now we may combine the two results.

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$$

$$= \begin{bmatrix} -1e^{2t} & 0 & 0 \\ -2e^{2t} & e^t(-\cos 2t + \sin 2t) & e^t(-\cos 2t - \sin 2t) \\ 1e^{2t} & e^t \cos 2t & e^t \sin 2t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} -4e^t \\ -3e^t \\ 5e^t \end{bmatrix}$$

The order of the columns of the fundamental matrix is arbitrary.

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Problem 3

a) (**12 points**)

Evaluate the line integral

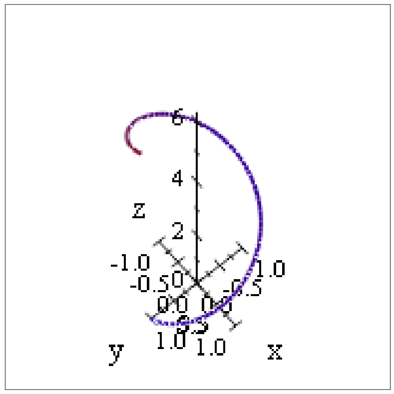
$$\int_C yze^{xyz}dx + xze^{xyz}dy + (xye^{xyz}+1)dz = \int_C \mathbf{F}.d\mathbf{r}$$

where C is one turn around a spiral defined by

$$x = \cos t, \quad y = \sin t, \quad z = t \quad 0 \le t \le 2\pi.$$

Here's a picture of C.

$(\cos t, \sin t, t)$



Solution: Direct substitution looks ugly. Let's check for path independence. That would give some options. We need to compute the curl of the integrand and hope to obtain zero.

$$\operatorname{curl} F =$$

$$= \mathbf{i}\left|\begin{bmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xze^{xyz} & xye^{xyz}+1 \end{bmatrix}\right| - \mathbf{j}\left|\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ yze^{xyz} & xye^{xyz}+1 \end{bmatrix}\right| + \mathbf{k}\left|\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ yze^{xyz} & xze^{xyz} \end{bmatrix}\right|$$

$$= \mathbf{i}\left(xe^{xyz} + x^2yze^{xyz} - xe^{xyz} - x^2yze^{xyz}\right)$$

$$- \mathbf{j}\left(ye^{xyz} + xy^2ze^{xyz} - ye^{xyz} - xy^2ze^{xyz}\right) + \mathbf{k}\left(ze^{xyz} + xyz^2e^{xyz} - ze^{xyz} - xyz^2e^{xyz}\right)$$

$$= 0$$

Hence the vector field is conservative and the integral is independent of the path. Let's use a straight (vertical) line from the initial point $(1,0,0)$ to the terminal point $(1,0,2\pi)$.

The parametrization and details are

$$x = 1, \quad y = 0, \quad z = t \qquad 0 \le t \le 2\pi$$

$$dx = 0 \quad dy = 0 \quad dz = dt$$

$$\int_C \mathbf{F}.d\mathbf{r} = \int_C yze^{xyz}dx + xze^{xyz}dy + (xye^{xyz}+1)dz$$

$$= \int_0^{2\pi} 1dt = 2\pi$$

Alternatively, direct substitution yields

$$\int_C \mathbf{F}.d\mathbf{r} = \int_0^{2\pi}\left[\left(-t\sin^2 t + t\cos^2 t + \cos t\sin t\right)e^{t\cos t\sin t} + 1\right]dt$$

Let

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$$u = t\cos t\sin t.$$

Then

$$du = -t\sin^2 t + t\cos^2 t + \cos t\sin t.$$

So the first term is $\int e^u du$, and we obtain

$$\int_C \mathbf{F}.d\mathbf{r} = \mathbf{e}^{t\cos t\sin t} + t|_0^{2\pi}$$

$$= 2\pi.$$

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b) (13 points)

Find the area of the surface of the part of the right circular cone $z^2 = x^2 + y^2$ which is between the planes $z = 0$ and $z = 1$.

Solution: The surface is the graph of a function, i.e. specified as

$$z = f(x,y) \qquad \text{for } (x,y) \text{ in a region } R.$$

The area of the surface is given by

$$A = \iint_R \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA$$

We haveπ

$$f(x,y) = \left(x^2 + y^2\right)^{\frac{1}{2}}$$

$$f_x = \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}}(2x)$$

$$f_y = \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}}(2y)$$

R is the unit circle.

$$A = \iint_{0 \le x^2 + y^2 \le 1} \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2}\, dA$$

$$= \iint_R \sqrt{2}\, dA = \sqrt{2}\,\pi$$

The integral is evaluated using the area of the unit circle.

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Problem 4

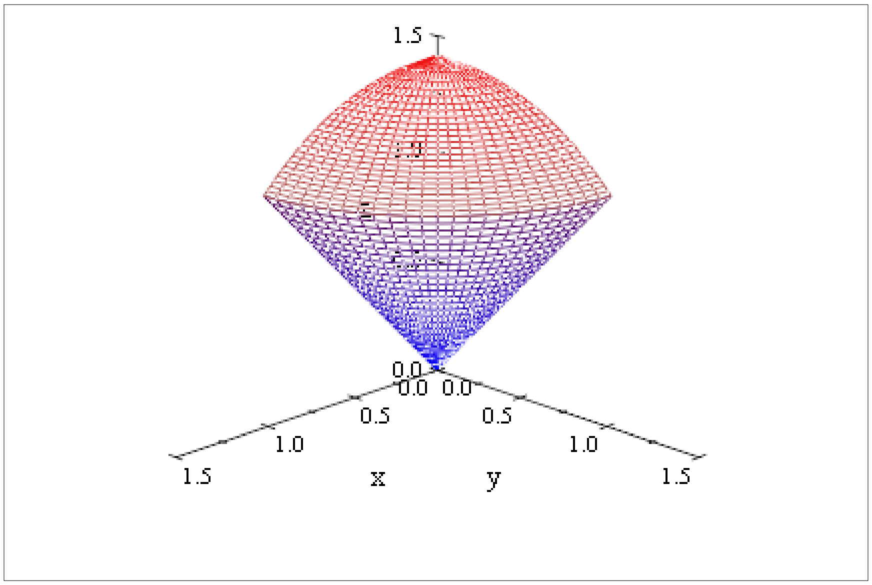
(25 points)
Consider the triple integral

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} (x^2+y^2+z^2)\,dzdydx.$$

a) Describe (in words and/or equations) and sketch the region of integration.

Solution: The region is in the first octant, bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+z^2 = 2$.

z



b). Give an equivalent triple integral in rectangular coordinates in a different order of integration.

Solution: Interchanging x and y is simple due to symmetry. The other four orders of integration are more complicated, since a sum of two integrals is required, one for the bottom part bounded by the cone and the other for the top portion bounded by the sphere. The simple case and one of the others are shown.

$$I = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} (x^2+y^2+z^2)\,dzdxdy$$

$$= \int_0^1\int_0^z \int_0^{\sqrt{z^2-y^2}} (x^2+y^2+z^2)\,dxdydz + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-z^2}} \int_0^{\sqrt{2-y^2-z^2}} (x^2+y^2+z^2)\,dxdydz$$

c). Give an equivalent triple integral in cylindrical coordinates.

Solution: This is straight forward. The inersection of the cone $(z^2 = x^2+y^2)$ and the sphere $(x^2+y^2+z^2 = 2)$ is $z = 1 = x^2+y^2$.

$$I = \int_0^{\frac{\pi}{2}} \int_0^1 \int_r^{\sqrt{1-r^2}} (r^2+z^2)\,rdzdrd\theta.$$

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$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \left(x^2+y^2+z^2\right) dzdydx.$$

c). Give an equivalent triple integral in spherical coordinates.

Solution: Since the equation of the cone in spherical coordinates is $\phi = \frac{\pi}{4}$ and the sphere is $\rho = \sqrt{2}$, this is the simplest of the integrals.

$$I = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} \left(\rho^2\right) \rho^2 \sin\phi \, d\rho d\phi d\theta.$$

e). Use any of your equivalent triple integrals to evaluate the integral.

Solution: The integral in spherical coordinates seems simplest.

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} \rho^4 \sin\phi \, d\rho d\phi d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \left[\frac{\rho^5}{5} \Big|_{\rho=0}^{\rho=\sqrt{2}} \right] \sin\phi \, d\phi d\theta \\
&= \frac{4\sqrt{2}}{5} \int_0^{\frac{\pi}{2}} \left[-\cos\phi \Big|_{\phi=0}^{\phi=\frac{\pi}{4}} \right] d\theta \\
&= \left(\frac{4\sqrt{2}}{5}\right)\left(\frac{-1}{\sqrt{2}} + 1\right)\left(\frac{\pi}{2}\right) \\
&= \frac{4\left(\sqrt{2}-1\right)}{10}\pi
\end{aligned}$$

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Problem 5

a) (**15 points**)

Show that

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

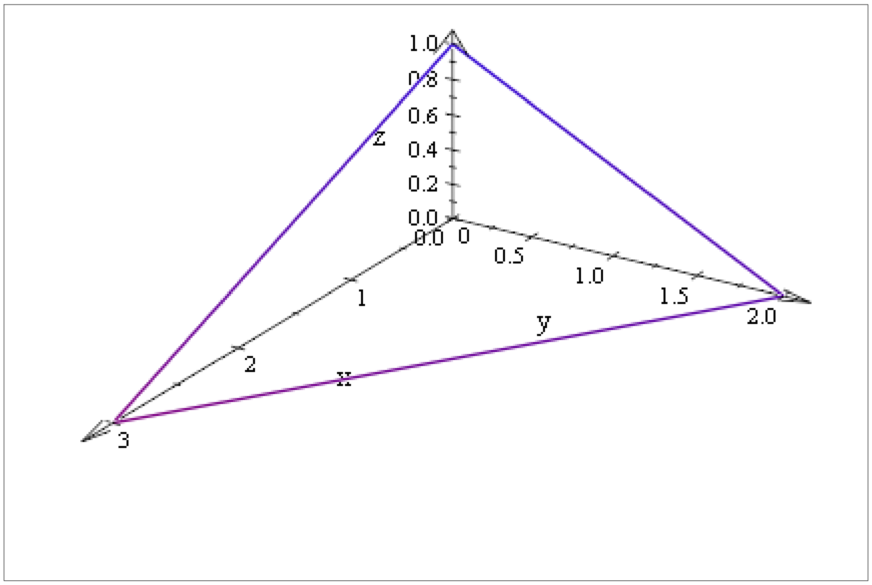
where

$$\vec{F} = \sin(x^2)\vec{i} + \left(e^{y^2} + x^2\right)\vec{j} + \left(z^4 + 2x^2\right)\vec{k}$$

and C consists of the line segment joining $(3,0,0)$ to $(0,2,0)$ followed by the line segment joining $(0,2,0)$ to $(0,0,1)$ followed by the line segment joining $(0,0,1)$ to $(3,0,0)$. Sketch C.

Solution: C is shown below

$(3,0,0,0,2,0,0,0,1,3,0,0)$



It is not possible to directly evaluate this line integral. However, we can apply Stokes Theorem, namely

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F} \cdot d\vec{S}$$

[Since the equation of the plane was given the discussion below is not required as part of the solution. The surface S is the plane passing through the three given points. The equation of a plane is

$$ax + by + cz = d$$

Substituting the point $(3,0,0)$ yields $d = 3a$, the point $(0,2,0)$ yields $d = 2b$, and the point $(0,0,1)$ yields $d = c$. Hence

$$\frac{d}{3}x + \frac{d}{2}y + dz = d$$

or

$$\frac{x}{3} + \frac{y}{2} + z = 1$$

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is the equation of the plane which is *S*.]

To find a normal to *S* we let

$$x = x,\ \ y = y,\ z = 1 - \frac{x}{3} - \frac{y}{2}$$

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + \left(1 - \frac{x}{3} - \frac{y}{2}\right)\vec{k}$$

$$\vec{r}_x = \vec{i} - \frac{1}{3}\vec{k}$$

$$\vec{r}_y = \vec{j} - \frac{1}{2}\vec{k}$$

Thus

$$\vec{r}_x \times \vec{r}_y = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{2} \end{matrix}\right|\begin{matrix} \vec{i} & \vec{j} \\ 1 & 0 \\ 0 & 1 \end{matrix} = \vec{k} + \frac{1}{3}\vec{i} + \frac{1}{2}\vec{j}$$

This is outer.

$$\operatorname{curl}\vec{F} = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin\left(x^2\right) & e^{y^2} + x^2 & z^4 + 2x^2 \end{matrix}\right|\begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \sin\left(x^2\right) & e^{y^2} + x^2 \end{matrix}$$

$$= 0\vec{i} + 0\vec{j} + 2x\vec{k} - 0\vec{k} - 0\vec{i} - 4x\vec{j}$$

Thus

$$\operatorname{curl}\vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = \left(-4x\vec{j} + 2x\vec{k}\right) \cdot \left(\frac{1}{3}\vec{i} + \frac{1}{2}\vec{j} + \vec{k}\right) = -2x + 2x = 0$$

Hence

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = 0$$

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b) (**10 points**)
Evaluate

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$$

where

$$\vec{F} = \left(x^2 + xz\right)\vec{i} + 2y^2\vec{j} - \frac{z^2}{2}\vec{k}$$

and S is surface of the cube bounded by $x = 0, x = 1. y = 0, y = 1, z = 0, z = 1.$
Solution: We use the divergence theorem

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int_S \vec{F} \cdot \vec{n}\, ds = \int\int\int_V \operatorname{div}\vec{F}\, dV = \int\int\int_V \nabla \cdot \vec{F}\, dV$$

Solution:

$$\operatorname{div}\vec{F} = 2x + z + 4y - z = 2x + 4y$$

Thus

$$\iint_S \vec{F} \cdot d\vec{S} = \int\int\int_V \operatorname{div}\vec{F}\, dV = \iiint_V (2x + 4y)\, dV$$

$$= \int_0^1\int_0^1\int_0^1 (2x + 4y)\, dz dx dy$$

$$= \int_0^1\int_0^1 (2x + 4y)\, dx dy$$

$$= \int_0^1 \left(x^2 + 4xy\right)_{x=0}^{x=1} dy = \int_0^1 (1 + 4y)\, dy$$

$$= \left(y + 2y^2\right)_0^1 = 3$$

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Problem 6

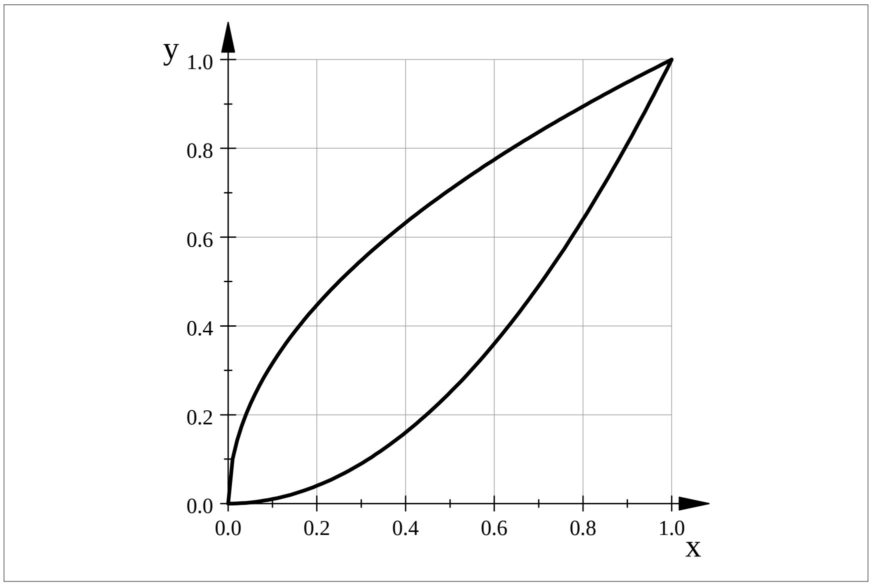
Verify Green's Theorem for the line integral

$$\oint_C \left(4x^3 + 2y^2\right)dx + (4xy + e^y)dy$$

where C is the boundary of the region between $y = x^2$ and $y = \sqrt{x}$. Sketch C.
Solution: Green's theorem is

$$\oint_C P(x,y)dx + Q(x,y)dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$$

C is the closed curve show below consisting of $y = x^2$ followed by $y = \sqrt{x}$ traversed conterclockwise.
R is the region enclosed by C.



$P = 4x^3 + 2y^2$ and $Q = 4xy + e^y$ so

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4y - 4y = 0$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = 0$$

Now we evaluate the line integral

$$\oint_C \left(4x^3 + 2y^2\right)dx + (4xy + e^y)dy$$

$$= \int_{\text{along }\ y=x^2} \left(4x^3 + 2y^2\right)dx + (4xy + e^y)dy + \int_{\text{along } y=\sqrt{x}} \left(4x^3 + 2y^2\right)dx + (4xy + e^y)dy$$

Along $y = x^2$ $\ dy = 2x$ whereas along $y = \sqrt{x}$ $\ dy = \frac{dx}{2\sqrt{x}}$. Thus

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$$\int_{\text{along } y=x^2} \left(4x^3 + 2y^2\right)dx + (4xy + e^y)dy + \int_{\text{along } y=\sqrt{x}} \left(4x^3 + 2y^2\right)dx + (4xy + e^y)dy$$

$$= \int_0^1 \left(4x^3 + 2x^4\right)dx + \left(4x^3 + e^{x^2}\right)2xdx + \int_1^0 \left(4x^3 + 2x\right)dx + \left(4x\sqrt{x} + e^{\sqrt{x}}\right)\frac{dx}{2\sqrt{x}}$$

$$= \int_0^1 \left(10x^4 + 4x^3 + 2xe^{x^2}\right)dx + \int_1^0 \left(4x^3 + 4x + \frac{e^{\sqrt{x}}}{2\sqrt{x}}\right)dx$$

$$= \left(2x^5 + x^4 + e^{x^2}\right)_{x=0}^{x=1} + \left(x^4 + 2x^2 + e^{\sqrt{x}}\right)_{x=1}^{x=0}$$

$$= 3 + e - 1 + 1 - 3 - e = 0$$

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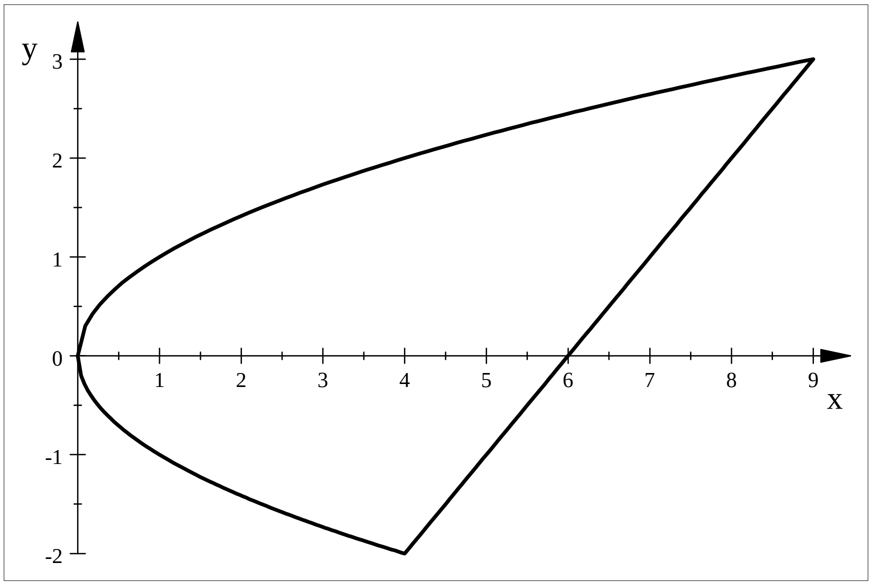
Problem 7

a) (**13 points**)

Give two double integral expressions for the area of the region R bounded by $y = x - 6$ and $y^2 = x$. Be sure to sketch R. Do *not* evaluate these expressions.

Solution: The region R is shown below.

$\sqrt{x}$



We will integrate first with respect to x and then with respect to y. We are given that $y^2 = x$. Since $y = x - 6$, this implies that $x = y + 6$. Next, we find the point of intersection of these two functions by setting the two functions equal to one another. We have $y^2 = y + 6$ or $y^2 - y - 6 = (y-3)(y+2) = 0$. Solving for y, we find that $y = -2$ and $y = 3$. Therefore the curves intersect at $(4,-2)$ and at $(9,3)$. The area of the region R is therefore given by

$$\text{Area } R = \int_{-2}^{3} \int_{y^2}^{y+6} dxdy$$

Now we will integrate first with respect to y and then with respect to x. We are given that $y = x - 6$. Since $y^2 = x$, we know that $y = \pm\sqrt{x}$. We need two integrals to express the area now. For $0 \le x \le 4$ y goes from the bottom of the parabola to the top of the parabola, whereas for $4 \le x \le 9$ y goes from the line to the top of the parabola. Thus

$$\text{Area } R = \int_{0}^{4} \int_{-\sqrt{x}}^{\sqrt{x}} dydx + \int_{4}^{9} \int_{x-6}^{\sqrt{x}} dydx$$

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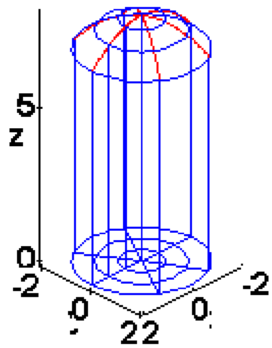
b) (**12 points**)
Give the expression in *cylindrical* coordinates for the volume of the solid inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$. Sketch the part of the volume in the first octant. Do *not* evaluate this expression.

SOLUTION



The ellipsoid intersects the x, y-plane in the circle $x^2 + y^2 = 16$. Thus, our region is bounded by the circle $x^2 + y^2 = 4$. So , in polar coordinates we have the equation $r = 2$. Next, we can solve the equation of the ellipsoid $4x^2 + 4y^2 + z^2 = 64$ for z, i.e., $z = \pm 2\sqrt{-x^2 - y^2 + 16}$ which can be rewritten in polar coordinates as $z = \pm 2\sqrt{16 - r^2}$. The volume of the solid can now be written as:

$$2\int_0^{2\pi}\int_0^2\int_0^{+2\sqrt{16-r^2}} r\,dz\,dr\,d\theta$$

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Problem 8

a) (**13 points**)

Let

$$A = \begin{bmatrix} -2 & -1 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & -3 \end{bmatrix}$$

Find A^{-1}.

Solution: We form

$$\begin{bmatrix} -2 & -1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 1 & -3 & 0 & 0 & 1 \end{bmatrix}$$

and row reduce. Adding R_1 to R_2 and to R_3 yields

$$\begin{bmatrix} -2 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & -2 & 1 & 0 & 1 \end{bmatrix}$$

Adding $2R_3$ to R_1 gives

$$\begin{bmatrix} 0 & -1 & -3 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & -2 & 1 & 0 & 1 \end{bmatrix}$$

Interchanging R_3 with R_1 gives

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & -3 & 3 & 0 & 2 \end{bmatrix}$$

Interchanging R_2 and R_3 gives

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & -1 & -3 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$3R_3 + R_2$ and $2R_3 + R_1$ gives

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 2 & 1 \\ 0 & -1 & 0 & 6 & 3 & 2 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Finally $(-1)R_2$ gives

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$$\begin{bmatrix} 1 & 0 & 0 & 3 & 2 & 1 \\ 0 & 1 & 0 & -6 & -3 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Thus

$$A^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ -6 & -3 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

SNB check $\begin{bmatrix} -2 & -1 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & -3 \end{bmatrix}$, inverse: $\begin{bmatrix} 3 & 2 & 1 \\ -6 & -3 & -2 \\ 1 & 1 & 0 \end{bmatrix}$

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b) (**12 points**)
Rewrite the initial value problem for the system

$$x'(t) = Ax(t) + f(t) \qquad x(0) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -e^t & t^2 \end{bmatrix}$$

and

$$f(t) = \begin{bmatrix} 0 \\ 0 \\ \sin 2t \end{bmatrix}$$

as a single differential equation with initial conditions.
Solution:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The given system is

$$x'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -e^t & t^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin 2t \end{bmatrix}$$

Let

$$x_1(t) = y(t), \; x_2(t) = y'(t), \; x_3(t) = y''(t)$$

so

$$x_1'(t) = y'(t) = x_2(t)$$
$$x_2'(t) = y''(t) = x_3(t)$$
$$x_3'(t) = y'''(t) = -3x_1 - e^t x_2 + t^2 x_3 + \sin 2t = -t^2 y'' - e^t y' - 3y + \sin 2t$$

Hence the differential equation is

$$y''' - t^2 y'' + e^t y' + 3y = \sin 2t$$

The the initial condition is

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix}$$

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so $y(0) = 2, y'(0) = -1, y''(0) = 0.$

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Table of Integrals

$$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$$

$$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$$

$$\int \sin^3 x dx = \frac{1}{12}\cos 3x - \frac{3}{4}\cos x + C$$

$$\int \cos^3 x dx = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + C$$

$$\int \left(\cos^2 x - \sin^2 x\right) dx = \frac{1}{2}\sin 2x + C$$