Ma 227 Final Exam Solutions 13 May 1997

Directions: This examination is in two parts. In Part I you must solve all six (6) problems. In Part II choose any two problems. Each problem is worth 25 points. If you need more work space, continue the problem you are doing on the other side of the page it is on.

Score on Problem #1 ________
#2 ________
#3 ________
#4 ________
#5 ________
#6 ________
#7 ________
#8 ________
#9 ________
Total ________
You may not use a calculator on this exam.

**Part I:** Answer all questions.

**Problem 1**

a) (8 points)
Find the first four nonzero terms of the Fourier sine series of

\[ f(x) = \begin{cases} 
0 & 0 < x < \frac{\pi}{2} \\
-2 & \frac{\pi}{2} < x < \pi 
\end{cases} \]

b) (8 points)
Sketch the graph of the function to which the Fourier series in (a) converges on \(-2\pi < x < 3\pi\).

c) (9 points)
Find the eigenvalues and eigenfunctions for the problem

\[ y'' + \lambda y = 0 ; \quad y'(0) = 0 ; \quad y\left(\frac{\pi}{2}\right) = 0 \]

Be sure to check the cases \( \lambda < 0, \lambda = 0, \) and \( \lambda > 0. \)
Problem 2

a) (10 points)
Use separation of variables, \( u(x, t) = X(x)T(t) \), to find ordinary differential equations which \( X(x) \) and \( T(t) \) must satisfy if \( u(x, t) \) is to be a solution of

\[
5t^4x^3 u_{xt} + (x + 2)^2(t - 5)u_{xx} = 0
\]

Do not solve these equations.

b) (15 points)
Solve:

P.D.E.: \( u_{xx} - 16u_{tt} = 0 \)

B.C.’s: \( u(0, t) = 0 \quad u_x(1, t) = 0 \)

I.C.: \( u(x, 0) = -3 \sin \frac{5x}{2} + 23 \sin \frac{11x}{2}; u_t(x, 0) = 0 \)
Problem 3

a) (15 points)
Find the eigenvalues and eigenvectors of
\[ A = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix} \]

b) (10 points)
Find the solution, if it exists, of
\[
\begin{align*}
    x_1 & - x_2 + 2x_3 = -1 \\
    x_3 & = 0 \\
    3x_1 & - 3x_2 + 7x_3 = 1 \\
    10x_1 & -10x_2 + 24x_3 = -2 
\end{align*}
\]
Problem 4

a) (10 points)
Let \( \mathbf{F} = -y \mathbf{i} + x \mathbf{j} - xyz \mathbf{k} \). Calculate \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( C \) is the curve \( x^2 + y^2 = 9, z = 3 \)

Solution:
Parametrize \( C \):
\[
x = 3 \cos \theta, \quad y = 3 \sin \theta, \quad z = 3 \quad 0 \leq \theta \leq 2\pi
\]
\[
\mathbf{r}(\theta) = 3 \cos \theta \mathbf{i} + 3 \sin \theta \mathbf{j} + 3 \mathbf{k}
\]
\[
\mathbf{r}'(\theta) = -3 \sin \theta \mathbf{i} + 3 \cos \theta \mathbf{j}
\]
\[
\mathbf{F}(\theta) = -3 \sin \theta \mathbf{i} + 3 \cos \theta \mathbf{j} - 27 \cos \theta \sin \theta \mathbf{k}
\]
\[
\mathbf{F}(\theta) \cdot \mathbf{r}''(\theta) = 9
\]
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 9 d\theta = 18\pi
\]

b) (13 points)
Using the \( \mathbf{F} \) in part a) calculate \( \iint_S \text{curl} \mathbf{F} \cdot \mathbf{n} \, ds \), where \( S \) is the surface of the cone \( z = \sqrt{x^2 + y^2} \) for \( x^2 + y^2 \leq 9 \)
(Note: \( \cos^2 \theta - \sin^2 \theta = \cos 2\theta \))

c) (2 points)
Did you get the same answer in parts a) and b)? If yes, why and if no, why not?
Problem 5

Consider the \( \int_C (6xy - 4e^x)dx + 3x^2dy \), where \( C \) is any piecewise smooth curve joining \((0,0)\) and \((-2,1)\)

a) \(10 \text{ points}\)

Show that this line integral is independent of path.

Solution:

\[ P = 6xy - 4e^x \]
\[ Q = 3x^2 \]

Thus

\[ Q_x = 6x = P_y \]

and the line integral is path independent.

b) \(15 \text{ points}\)

Calculate the line integral given above.

Solution:

We find a potential function \( G(x,y) \)

\[ G_x = P = 6xy - 4e^x \]

so

\[ G = 3x^2y - 4e^x + h(y) \]

Then

\[ G_y = 3x^2 + h'(y) = Q = 3x^3 \]

so \( h(y) = K \). Thus

\[ G(x,y) = 3x^2y - 4e^x + K \]

\[ \int_C (6xy - 4e^x)dx + 3x^2dy = G(-2,1) - G(0,0) = 16 - 4e^{-2} \]

Problem 6

a) \(13 \text{ points}\)

Let \( S \) be the closed surface consisting of the cone \( z^2 = x^2 + y^2 \) for \( 0 \leq z \leq 1 \) and the “top” \( x^2 + y^2 \leq 1, z = 1 \). Let \( \vec{F} = (x + e^y)\hat{i} + y\hat{j} + z\hat{k} \). Use the divergence theorem to calculate \( \int_S \vec{F} \cdot \vec{n} \, ds \)
b) (12 points)
Give an expression for the volume inside the sphere $x^2 + y^2 + z^2 = 4$
and outside the cylinder $x^2 + y^2 = 1$
Part II: Choose any two questions.

Problem 7

a) (13 points)
Evaluate $\int \int_S \vec{v} \cdot \vec{n} dS$, where $\vec{v} = 2\vec{j} + xy\vec{k}$ and $S$ is the part of the paraboloid $2z = x^2 + y^2$ inside $x^2 + y^2 = 6x$, and $\vec{n}$ is the unit normal to $S$ such that $\vec{n} \cdot \vec{k} > 0$.

b) (12 points)
Find the volume between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$ bounded below by the $x,y-$plane and above by $z = 3y + 1$, where $y \geq 0$. 
Problem 8

a) (13 points)

Use Stokes’ Theorem to evaluate \( \int_C \mathbf{v} \cdot d\mathbf{r} \), where \( \mathbf{v} = y\mathbf{i} + xz\mathbf{j} + x^2\mathbf{k} \) and \( C \) is the boundary of the triangle cut from the plane \( x + y + z = 1 \) in the first octant, traversed counterclockwise when viewed from above.

b) (12 points)

Find all values of \( b \) for which

\[
\begin{vmatrix}
  b & 0 & 1 & b \\
  0 & 1 & b & b \\
  1 & b & 0 & 2 \\
  0 & 1 & 1 & b \\
\end{vmatrix} = (1 - b)(5 + b^3)
\]
Problem 9

a) (10 points)
Suppose that \( f(x) = \sum_{i=1}^{\infty} a_i \phi_i(x) \), where \( \{\phi_1, \phi_2, \phi_3, \ldots\} \) is an orthonormal set on the interval \([a, b]\). Show that \( \int_{a}^{b} f^2(x) \, dx = \sum_{1}^{\infty} a_i^2 \)

b.) (15 points)
Find the inverse of the matrix

\[
\begin{pmatrix}
1 & 3 & 2 \\
0 & 2 & 5 \\
0 & 0 & -1
\end{pmatrix}
\]