Name: SSN: Instructor:

Score on Final:________________________________

I pledge my honor that I have abided by the Stevens Honor System.____

Ma 227 Final Exam Solutions13 May 1997

Directions: This examination is in two parts. In Part I you must solve all six (6) problems. In Part II choose any two problems. Each problem is worth **25** points. If you need more work space, continue the problem you are doing on the **other side of the page it is on**.

Score on Problem #1 _________
#2 _________
#3 _________
#4 _________
#5 _________
#6 _________
#7 _________
#8 _________
#9 _________
Total _________

You may *not* use a calculator on this exam.

Part I: Answer **all** questions.

Problem 1

a) (8 points)

Find the first four nonzero terms of the Fourier *sine* series of

$$f(x) = \begin{cases} 0 & 0 < x < \frac{\pi}{2} \\ -2 & \frac{\pi}{2} < x < \pi \end{cases}$$

b) (8 points)

Sketch the graph of the function to which the Fourier series in (a) converges on $-2\pi < x < 3\pi$.

c) (9 points)

Find the eigenvalues and eigenfunctions for the problem

$$y'' + \lambda y = 0 \; ; \qquad y'(0) = 0 \; ; \qquad y(\tfrac{\pi}{2}) = 0$$

Be sure to check the cases $\lambda < 0, \lambda = 0,$ and $\lambda > 0.$

Problem 2

a) **(10 points)**

Use separation of variables, $u(x,t) = X(x)T(t)$, to find ordinary differential equations which $X(x)$ and $T(t)$ must satisfy if $u(x,t)$ is to be a solution of

$$5t^4x^3u_{xt} + (x+2)^2(t-5)u_{xx} = 0$$

Do *not* solve these equations.

b) **(15 points)**

Solve:

P.D.E.: $u_{xx} - 16u_{tt} = 0$

B.C.'s: $u(0,t) = 0 \qquad u_x(1,t) = 0$

I.C.: $u(x,0) = -3\sin\frac{5\pi x}{2} + 23\sin\frac{11\pi x}{2}; u_t(x,0) = 0$

Problem 3

a) (**15 points**)

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

b) (**10 points**)

Find the solution, if it exists, of

$$\begin{aligned} x_1 \quad - x_2 \quad + 2x_3 &= -1 \\ x_3 &= 0 \\ 3x_1 \quad - 3x_2 \quad + 7x_3 &= 1 \\ 10x_1 \quad - 10x_2 \quad + 24x_3 &= -2 \end{aligned}$$

Problem 4

a) (10 points)

Let $\vec{F} = -y\vec{i} + x\vec{j} - xyz\vec{k}$. Calculate $\oint_C \vec{F} \bullet d\vec{r}$, where C is the curve $x^2 + y^2 = 9, z = 3$

Solution:

Parametrize C :

$$x = 3\cos\theta,\ \ y = 3\sin\theta,\ \ z = 3 \quad 0 \le \theta \le 2\pi$$

$$\vec{r}(\theta) = 3\cos\theta\vec{i} + 3\sin\theta\vec{j} + 3\vec{k}$$

$$\vec{r}'(\theta) = -3\sin\theta\vec{i} + 3\cos\theta\vec{j}$$

$$\vec{F}(\theta) = -3\sin\theta\vec{i} + 3\cos\theta\vec{j} - 27\cos\theta\sin\theta\vec{k}$$

$$\vec{F}(\theta) \cdot \vec{r}'(\theta) = 9$$

$$\oint_C \vec{F} \bullet d\vec{r} = \int_0^{2\pi} 9d\theta = 18\pi$$

b) (13 points)

Using the $\vec{F}$ in part a) calculate $\iint_S curl\vec{F} \bullet \vec{n}ds$, where S is the surface of the cone $z = \sqrt{x^2 + y^2}$ for $x^2 + y^2 \le 9$

(Note: $\cos^2\theta - \sin^2\theta = \cos 2\theta$)

c) (2 points)

Did you get the same answer in parts a) and b) ? If yes, why and if no,why not?

Problem 5

Consider the $\int_C (6xy - 4e^x)dx + 3x^2 dy,$ where C is any piecewise smooth curve joining $(0,0)$ and $(-2,1)$

a) (10 points)

Show that this line integral is independent of path.

Solution:

$$P = 6xy - 4e^x$$
$$Q = 3x^2$$

Thus

$$Q_x = 6x = P_y$$

and the line integral is path independent.

b) (15 points)

Calculate the line integral given above.

Solution:

We find a potential function $G(x,y)$

$$G_x = P = 6xy - 4e^x$$

so

$$G = 3x^2y - 4e^x + h(y)$$

Then

$$G_y = 3x^2 + h'(y) = Q = 3x^3$$

so $h(y) = K.$ Thus

$$G(x,y) = 3x^2y - 4e^x + K$$

$$\int_C (6xy - 4e^x)dx + 3x^2dy = G(-2,1) - G(0,0) = 16 - 4e^{-2}$$

Problem 6

a) (13 points)

Let S be the closed surface consisting of the cone $z^2 = x^2 + y^2$ for $0 \le z \le 1$ and the “top” $x^2 + y^2 \le 1, z = 1.$ Let $\vec{F} = (x + e^{y^2})\vec{i} + y\vec{j} + z\vec{k}.$ Use the divergence theorem to calculate $\iint_S \vec{F} \bullet \vec{n} ds$

b) (**12 points**)

Give an expression for the volume inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $x^2 + y^2 = 1$

Part II: Choose any *two* questions.

Problem 7

a) **(13 points)**

Evaluate $\int\int_S \vec{v} \cdot \vec{n} dS$, where $\vec{v} = 2\vec{j} + xy\vec{k}$ and S is the part of the paraboloid $2z = x^2 + y^2$ inside $x^2 + y^2 = 6x$, and $\vec{n}$ is the unit normal to S such that $\vec{n} \cdot \vec{k} > 0$.

b) **(12 points)**

Find the volume between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$ bounded below by the x, y –plane and above by $z = 3y + 1$, where $y \geq 0$.

Problem 8

a) (**13 points**)

Use Stokes' Theorem to evaluate $\oint_C \vec{v} \bullet d\vec{r}$, where $\vec{v} = y\vec{i} + xz\vec{j} + x^2\vec{k}$ and C is the boundary of the triangle cut from the plane $x + y + z = 1$ in the first octant, traversed counterclockwise when viewed from above.

b) (**12 points**)

Find all values of b for which

$$\begin{vmatrix} b & 0 & 1 & b \\ 0 & 1 & b & b \\ 1 & b & 0 & 2 \\ 0 & 1 & 1 & b \end{vmatrix} = (1-b)(5+b^3)$$

Problem 9

a) (**10 points**)

Suppose that $f(x) = \sum_1^\infty a_i\phi_i(x),$ where $\{\phi_1,\phi_2,\phi_3,\dots\}$ is an orthonormal set on the interval $[a,b]$. Show that $\int_a^b f^2(x)dx = \sum_1^\infty a_i^2$

b.) (**15 points**)

Find the inverse of the matrix

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{pmatrix}$$