Review for MA227 Final

Chapter 12 - Multiple Integration

1. Evaluate the iterated integral: \( \int_{0}^{1} \int_{x}^{1} e^{y^3} dy \, dx \).

Solution:

We evaluate by switching the order of integration. The region in the xy-plane is above the curve \( y = \sqrt{x} \), below the line \( y = 1 \) and to the right of \( x = 0 \).

\[ y = \sqrt{x} \]

So

\[ \int_{0}^{1} \int_{x}^{1} e^{y^3} \, dy \, dx = \int_{0}^{1} \int_{0}^{y^2} e^{y^3} \, dx \, dy = \int_{0}^{1} x e^{y^3} \bigg|_{0}^{y^2} \, dy = \int_{0}^{1} y^2 e^{y^3} \, dy = \frac{1}{3} e^{y^3} \bigg|_{0}^{1} = \frac{1}{3} (e - 1) \]

(using \( u \)-substitution).

2.a) Sketch the region of integration: \( \int_{-2}^{2} \int_{x^2}^{4} x^2 \, y \, dy \, dx \).

Solution:
b) Reverse the order of integration.

Solution:

\[ y = x^2 \Rightarrow x = \pm \sqrt{y}, \quad 0 \leq y \leq 4, \text{ so} \]

\[ \int_{-2}^{2} \int_{x^2}^{4} x^2 \, dy \, dx = \int_{0}^{4} \int_{\sqrt{y}}^{2} x^2 \, dy \, dx. \]

3. Evaluate \( \iint_{D} y \, dy \, dx \), where \( D \) is the region in the first quadrant that lies above the hyperbola \( xy = 1 \) and the line \( y = x \) and below the line \( y = 2 \).

Solution:

For this region, it’s easiest to integrate first with respect to \( x \). We have \( \frac{1}{y} \leq x \leq y \), and \( 1 \leq y \leq 2 \), so

\[ \int_{1}^{2} \int_{\frac{1}{y}}^{y} y \, dx \, dy = \int_{1}^{2} y \left( y - \frac{1}{y} \right) \, dy = \int_{1}^{2} \left( y^2 - 1 \right) \, dy = \frac{y^3}{3} - y |_{1}^{2} = \frac{4}{3}. \]

If we integrate first with respect to \( y \), we have to break it into 2 regions:

\[ \frac{1}{2} \leq x \leq 1, \quad \frac{1}{x} \leq y \leq 2, \quad \text{and} \quad 1 \leq x \leq 2, x \leq y \leq 2 \]

\[ \Rightarrow \int_{1}^{2} \int_{\frac{1}{x}}^{y} y \, dx \, dy = \int_{\frac{1}{2}}^{1} \int_{1}^{\frac{1}{x}} y \, dy \, dx + \int_{1}^{2} \int_{x}^{2} y \, dy \, dx = \frac{4}{3}. \]

4. Consider \( \iiint_{\Delta} f(x,y) \, dx \, dy \).

a) Sketch the region of integration.

Solution:

\[ 0 \leq y \leq 2, y \leq x \leq 2 \]
b) Reverse the order of integration:
Solution:
\[0 \leq x \leq 2, 0 \leq y \leq x \Rightarrow \]
\[\int_{0}^{2} \int_{y}^{x} f(x,y) \, dx \, dy = \int_{0}^{2} \int_{0}^{x} f(x,y) \, dy \, dx\]

5. Evaluate \(\int_{D} x \sqrt{x^2 + y^2} \, dA\), where \(D\) is the closed disk with radius 1 and center \((0,1)\).

Solution:
Since we’re integrating over a circle it makes sense to use polar coordinates. The disk with radius 1 and center \((0,1)\) is given in rectangular coordinates by \(x^2 + (y-1)^2 = 1\). Convert to polar:
\[x^2 + (y-1)^2 = 1 \Rightarrow x^2 + y^2 - 2y + 1 = 1 \Rightarrow x^2 + y^2 = 2y \Rightarrow r^2 = 2r \sin \theta \Rightarrow r = 2 \sin \theta.\]
The entire circle is traced out as \(\theta\) ranges from 0 to \(\pi\), so we have
\[\int_{D} x \sqrt{x^2 + y^2} \, dA = \int_{0}^{\pi} \int_{0}^{2 \sin \theta} r \cos \theta \sqrt{r^2} \, r \, dr \, d\theta = \int_{0}^{\pi} \int_{0}^{2 \sin \theta} r^3 \cos \theta \, dr \, d\theta = \int_{0}^{\pi} \frac{r^4}{4} \cos \theta \bigg|_{0}^{2 \sin \theta} \, d\theta = \int_{0}^{\pi} 4 \sin^4 \theta \cos \theta \, d\theta = \frac{4}{5} \sin^5 \theta \bigg|_{0}^{\pi} = 0\]
(Using \(u\)-substitution)

6. Evaluate \(\int \int_{E} y \, dV\), where \(E\) is the tetrahedron bounded by the planes \(x = 0, \ y = 0, \ z = 0\) and \(2x + y + z = 2\).
Solution:

\[
2 - 2x - y
\]

In the xy-plane we have

\[
\int_0^1 \int_0^{2-2x} y \, dy \, dx
\]

where \(0 \leq y \leq 2 - 2x\), \(0 \leq x \leq 1\)

so \(\int \int_E y \, dV = \int_0^1 \int_0^{2-2x} y \, dy \, dx = \int_0^1 \int_0^{2-2x} \left. \frac{y^2}{2} \right|_0^{2-2x} \, dx\)

\[
= \int_0^1 \left. \frac{(2y - 2yx - y^2)}{2} \right|_0^{2-2x} \, dx = \int_0^1 \left( \frac{y^2 - y^2x - \frac{y^3}{3}}{2} \right) \, dx
\]

\[
= \left. \frac{y^3}{3} \right|_0^{2-2x} = \frac{2^3}{3} - \frac{y^3}{3} \bigg|_0^1 = \frac{8}{3} - \frac{1}{3} = \frac{1}{3}.
\]

7. Give an expression for the volume inside the sphere \(x^2 + y^2 + z^2 = 4\) and outside the cylinder \(x^2 + y^2 = 1\).

Solution:
We’ll consider half of the figure and then multiply our answer by 2. Take the top half of the figure. Then we have \( 0 \leq z \leq \sqrt{4 - x^2 - y^2} \). The region over which we integrate in the \( xy \)-plane is the annular region between the circles \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \):

\[
\begin{align*}
V &= 2 \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} rdzdrd\theta \\
&= 2\pi \int_0^1 (4-r^2) rdr \\
&= 2\pi \left[ \frac{4r^2}{2} - \frac{r^4}{4} \right]_0^1 \\
&= 2\pi \left( 2 - \frac{1}{4} \right) \\
&= \frac{15}{2} \pi.
\end{align*}
\]

8. Find the volume of the region of the ball \( x^2 + y^2 + (z-1)^2 = 1 \) cut out by the cone \( z^2 = x^2 + y^2 \) using spherical coordinates.

Solution:
Recall that in spherical coordinates, \( x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi, \)
\[ dV = \rho^2 \sin \phi. \]
\[ \Rightarrow x^2 + y^2 + (z - 1)^2 = 1 \Rightarrow (\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 + (\rho \cos \phi - 1)^2 = 1 \]
\[ \Rightarrow \rho^2 = 2 \rho \cos \phi \Rightarrow \rho = 2 \cos \phi. \]

The equation of the cone becomes \((\rho \cos \phi)^2 = (\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2\)
\[ \Rightarrow \tan^2 \phi = 1 \Rightarrow \phi = \frac{\pi}{4}. \]

So

\[ V = \int_{0}^{2\pi} \int_{0}^{\frac{\pi}{4}} \int_{0}^{2 \cos \phi} \rho^2 \sin \phi d\rho d\phi d\theta = \pi \]

9. Find the volume of the solid bounded by the plane \( z = 0 \) and the paraboloid \( z = 1 - x^2 - y^2. \)
Sketch the volume.

Solution:

The paraboloid \( z = 1 - x^2 - y^2 \) intersects the \( x, y \)–plane on the circle \( x^2 + y^2 = 1. \) Let \( D \) denote the inside of the circle. Then the volume is
\[ V = \iiint_D \int_0^{1-x^2-y^2} dz \, dA \]

Using cylindrical coordinates \( x = r \cos \theta, \ y = r \sin \theta, \ z = z \) we have,

\[ V = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (1 - r^2) \, r \, dr \, d\theta = \frac{\pi}{2} \]