Review for MA227 Final

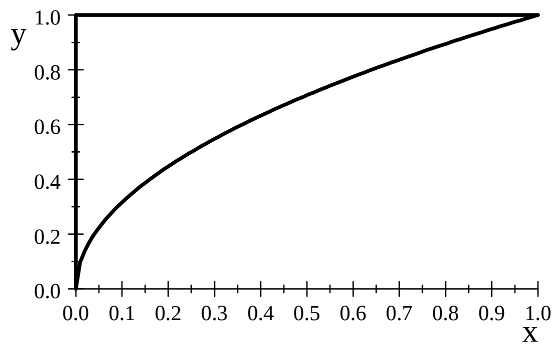
Chapter 12 - Multiple Integration

1. Evaluate the iterated integral: $\int_0^1 \int_{\sqrt{x}}^1 e^{y^3} dydx$.

Solution:

We evaluate by switching the order of integration. The region in the *xy*-plane is above the curve $y = \sqrt{x}$, below the line $y = 1$ and to the right of $x = 0$.

$y = \sqrt{x}$



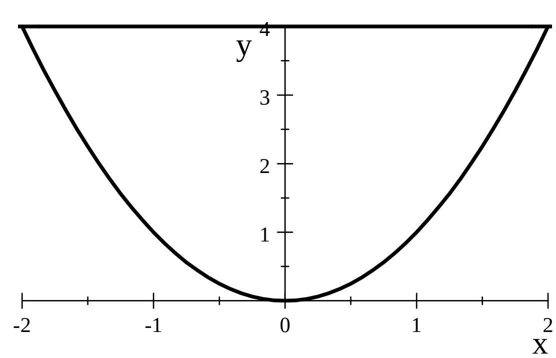
So

$$\int_0^1 \int_{\sqrt{x}}^1 e^{y^3} dydx = \int_0^1 \int_0^{y^2} e^{y^3} dxdy = \int_0^1 xe^{y^3}|_0^{y^2} dy = \int_0^1 y^2 e^{y^3} dy = \frac{1}{3} e^{y^3}|_0^1 = \frac{1}{3}(e-1)$$

(using *u*-substitution).

2.a) Sketch the region of integration: $\int_{-2}^2 \int_{x^2}^4 x^2 ydydx$.

Solution:



b) Reverse the order of integration.

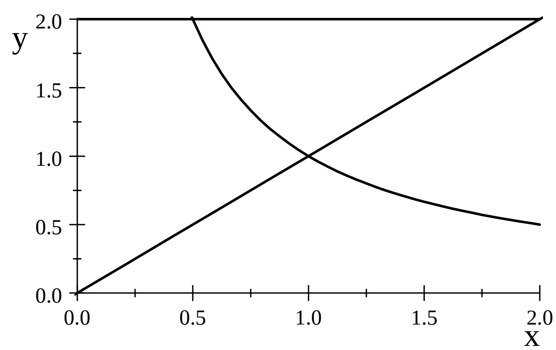
Solution:

$$y = x^2 \Rightarrow x = \pm\sqrt{y},\ 0 \le y \le 4, \text{ so}$$

$$\int_{-2}^{2}\int_{x^2}^{4} x^2 y\,dy\,dx = \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}} x^2 y\,dx\,dy.$$

3. Evaluate $\iint_D y\,dA$, where D is the region in the first quadrant that lies above the hyperbola $xy = 1$ and the line $y = x$ and below the line $y = 2$.

Solution:



$\frac{1}{x}$

For this region, it's easiest to integrate first with respect to x. We have $\frac{1}{y} \le x \le y$, and $1 \le y \le 2$, so

$$\iint_D y\,dA = \int_1^2\int_{\frac{1}{y}}^{y} y\,dx\,dy = \int_1^2 y\left(y - \frac{1}{y}\right)dy = \int_1^2 \left(y^2 - 1\right)dy = \frac{y^3}{3} - y\Big|_1^2 = \frac{4}{3}.$$

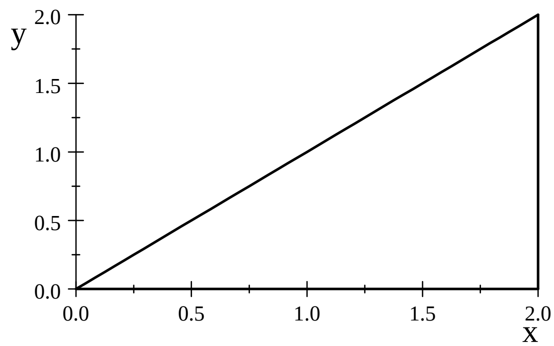
If we integrate first with respect to y, we have to break it into 2 regions:
$\frac{1}{2} \le x \le 1,\ \frac{1}{x} \le y \le 2,$ and $1 \le x \le 2, x \le y \le 2$
$\Rightarrow$

$$\iint_D y\,dA = \int_{\frac{1}{2}}^{1}\int_{\frac{1}{x}}^{2} y\,dy\,dx + \int_1^2\int_x^2 y\,dy\,dx = \frac{4}{3}$$

4. Consider $\int_0^2\int_y^2 f(x,y)\,dx\,dy$.

a) Sketch the region of integration.
Solution:

$0 \le y \le 2, y \le x \le 2$



b) Reverse the order of integration:

Solution:

$$0 \le x \le 2, 0 \le y \le x \Rightarrow$$
$$\int_0^2 \int_y^2 f(x,y)dxdy = \int_0^2 \int_0^x f(x,y)dydx$$

c) Rewrite the integral in terms of polar coordinates.

Solution:

In polar coordinates, the line $y = x$ is given by $\theta = \frac{\pi}{4}$. Noting that the polar equation of the line $x = 2$ is $r\cos\theta = 2$ or $r = 2\sec\theta$, we have

$$\int_0^{\pi/4} \int_0^{2\sec\theta} f(r\cos\theta, r\sin\theta) r dr d\theta.$$

5. Evaluate $\iint_D x\sqrt{x^2+y^2}\, dA$, where D is the closed disk with radius 1 and center $(0,1)$.

Solution:

Since we're integrating over a circle it makes sense to use polar coordinates. The disk with radius 1 and center $(0,1)$ is given in rectangular coordinates by $x^2 + (y-1)^2 = 1$. Convert to polar:

$$x^2 + (y-1)^2 = 1 \Rightarrow x^2 + y^2 - 2y + 1 = 1 \Rightarrow x^2 + y^2 = 2y$$
$$\Rightarrow r^2 = 2r\sin\theta \Rightarrow r = 2\sin\theta.$$

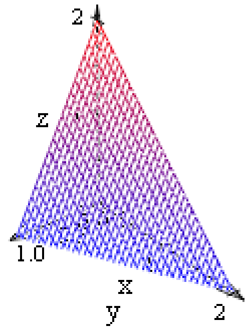
The entire circle is traced out as θ ranges from 0 to π, so we have

$$\iint_D x\sqrt{x^2+y^2}\, dA = \int_0^\pi \int_0^{2\sin\theta} r\cos\theta\sqrt{r^2}\, r dr d\theta = \int_0^\pi \int_0^{2\sin\theta} r^3 \cos\theta dr d\theta = \int_0^\pi \frac{r^4}{4}\cos\theta\Big|_0^{2\sin\theta} d\theta$$
$$= \int_0^\pi 4\sin^4\theta\cos\theta d\theta = \frac{4}{5}\sin^5\theta\Big|_0^\pi = 0$$

(using u-substitution)

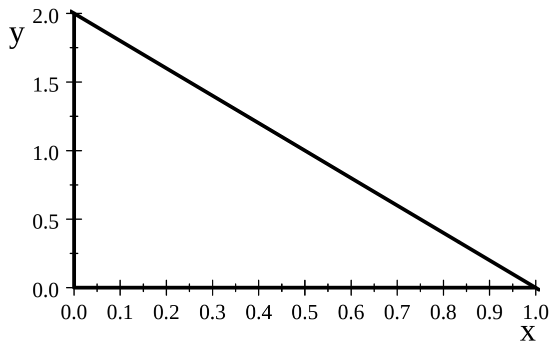
6. Evaluate $\iiint_E y dV$, where E is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + y + z = 2$.

Solution:



$2 - 2x - y$

In the *xy*-plane we have



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where $0 \le y \le 2 - 2x, 0 \le x \le 1$

so $\int\int\int_E ydV = \int_0^1 \int_0^{2-2x} \int_0^{2-2x-y} ydzdydx = \int_0^1 \int_0^{2-2x} yz|_0^{2-2x-y} dydx$

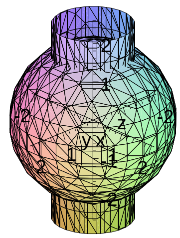
$$= \int_0^1 \int_0^{2-2x} \left(2y - 2yx - y^2\right) dydx = \int_0^1 \left(y^2 - y^2x - \frac{y^3}{3}\right)|_0^{2-2x} dx$$

$$= \int_0^1 \left(\frac{4}{3} - 4x + 4x^2 - \frac{4}{3}x^3\right) dx = \frac{4}{3}x - 2x^2 + \frac{4}{3}x^3 - \frac{x^4}{3}|_0^1$$

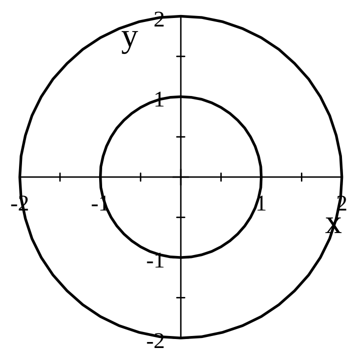
$$= \frac{4}{3} - 2 + \frac{4}{3} - \frac{1}{3} = \frac{1}{3}.$$

7. Give an expression for the volume inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $x^2 + y^2 = 1$.

Solution:



We'll consider half of the figure and then multiply our answer by 2. Take the top half of the figure. Then we have $0 \le z \le \sqrt{4 - x^2 - y^2}$. The region over which we integrate in the *xy*-plane is the annular region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$:



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We'll use cylindrical coordinates to find the volume. Recall that in cylindrical coordinates,
$x = r\cos\theta, y = r\sin\theta, z = z,$ and
$dV = rdzdrd\theta$. Our z limits become $0 \le z \le \sqrt{4 - r^2}$, and we have $1 \le r \le 2, \quad 0 \le \theta \le 2\pi$
$\Rightarrow$

$$V = 2\int_0^{2\pi}\int_1^2\int_0^{\sqrt{4-r^2}} rdzdrd\theta$$

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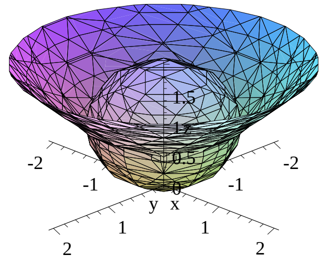
8. Find the volume of the region of the ball $x^2 + y^2 + (z-1)^2 = 1$ cut out by the cone $z^2 = x^2 + y^2$ using spherical coordinates.

Solution:



Recall that in spherical coordinates, $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$, $dV = \rho^2\sin\phi$.

$\Rightarrow x^2 + y^2 + (z-1)^2 = 1 \Rightarrow (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 + (\rho\cos\phi - 1)^2 = 1$

$\Rightarrow \rho^2 = 2\rho\cos\phi \Rightarrow \rho = 2\cos\phi$.

The equation of the cone becomes $(\rho\cos\phi)^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$

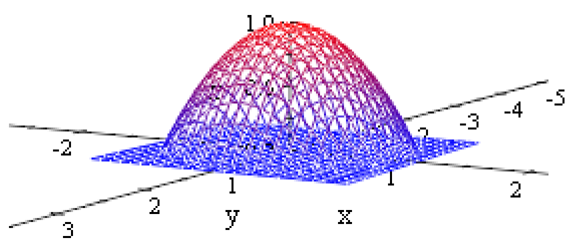
$$\Rightarrow \tan^2\phi = 1 \Rightarrow \phi = \frac{\pi}{4}.$$

So

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2\cos\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \pi$$

9. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$. Sketch the volume.

Solution:



$1 - x^2 - y^2$

The paraboloid $z = 1 - x^2 - y^2$ intersects the x, y –plane on the circle $x^2 + y^2 = 1$. Let D denote the inside of the circle. Then the volume is

$$V = \iint_D \int_0^{1-x^2-y^2} dz dA$$

Using cylindrical coordinates $x = r\cos\theta,\ y = r\sin\theta,\ z = z$ we have,

$$V = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\ dzdrd\theta = \int_0^{2\pi}\int_0^1 \left(1 - r^2\right) rdrd\theta = \frac{\pi}{2}$$