Ma 227 Review of Surface Integrals, Stokes' Theorem, and Divergence Theorem

Surface Integrals

Suppose $f(x,y,z)$ is a function of three variables whose domain includes a surface S. Then

$$\iint_S f(x,y,z)ds$$

is called the surface integral of f over S. Suppose that a surface S has a vector equation

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

where the parameters (u,v) have values in some domain D. We define the derivatives (tangent vectors)

$$\vec{r}_u(u,v) = \frac{\partial x(u,v)}{\partial u}\vec{i} + \frac{\partial y(u,v)}{\partial u}\vec{j} + \frac{\partial z(u,v)}{\partial u}\vec{k}$$

and

$$\vec{r}_v(u,v) = \frac{\partial x(u,v)}{\partial v}\vec{i} + \frac{\partial y(u,v)}{\partial v}\vec{j} + \frac{\partial z(u,v)}{\partial v}\vec{k}$$

If $\vec{r}_u$ and $\vec{r}_v$ are nonzero and non-parallel in D, the surface integral over S is given as

$$\iint_S f(x,y,z)ds = \iint_G f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|dudv$$

where G is the image of the surface S in the u,v –plane, and $f(\vec{r}(u,v))$ is short for $f(x(u,v),y(u,v),z(u,v))$.

Graph of $z = g(x,y)$

Any surface with equation

$$z = g(x,y)$$

can be regarded as a parametric surface with parametric equations

$$x = u, \quad y = v, \quad z = g(u,v)$$

that is,

$$\vec{r}(u,v) = u\vec{i} + v\vec{j} + g(u,v)\vec{k}$$

Now

$$\vec{r}_u = \vec{i} + g_u(u,v)\vec{k}$$
$$\vec{r}_v = \vec{j} + g_v(u,v)\vec{k}$$

so that

$$\vec{r}_u \times \vec{r}_v = -g_u\vec{i} - g_v\vec{j} + \vec{k}$$

and

$$|\vec{r}_u \times \vec{r}_v| = [1 + g_u^2 + g_v^2]^{\frac{1}{2}}$$

Because $u = x, v = y$ we get

$$\iint_S f(x,y,z)ds = \iint_G f(x,y,g(x,y))[1 + g_x^2 + g_y^2]^{\frac{1}{2}} dxdy$$

Hence, if we let $f(x,y,z) = 1,$ we get

$$\iint_S f(x,y,z)ds = \iint_G [1 + g_x^2 + g_y^2]^{\frac{1}{2}} dxdy$$

the area of $S,$ as we should.
In general

$$\iint_S f(x,y,z)ds = \iint_G f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v| dudv$$

gives for $f(x,y,z) = 1$ the area of S.

In class we evaluated $\iint_S f(x,y,z)ds$ where $f = x^2$ and S was the part of the cone $z = x^2 + y^2$ between the planes $z = 1$ and $z = 2.$ We used spherical coordinates and set $\phi = \frac{\pi}{4}$ for the equation of the cone. Here, let us do a somewhat simplified example.
Example 1: Compute the surface integral $\iint_S x^2 ds$ where S is the unit sphere $x^2 + y^2 + z^2 = 1.$

Solution:
We use spherical coordinates, $\rho = 1;\ \ 0 \le \phi \le \pi,\ \ 0 \le \theta \le 2\pi$

$$x = \sin\phi\cos\theta,\ \ y = \sin\phi\sin\theta,\ \ z = \cos\phi$$

$$\vec{r}(\phi,\theta) = \sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k}$$

$$\vec{r}_\phi = \cos\phi\cos\theta\vec{i} + \cos\phi\sin\theta\vec{j} - \sin\phi\vec{k}$$

$$\vec{r}_\theta = -\sin\phi\sin\theta\vec{i} + \sin\phi\cos\theta\vec{j}$$

so

$$\vec{r}_\varphi \times \vec{r}_\theta = \sin^2\phi\cos\theta\vec{i} + \sin^2\phi\sin\theta\vec{j} + \sin\phi\cos\phi\vec{k}$$

$$|\vec{r}_\varphi \times \vec{r}_\theta| = \sin\phi$$

Therefore,

$$\iint_S x^2 ds = \int_0^{2\pi}\int_0^{\pi}(\sin\phi\cos\theta)^2|\vec{r}_\varphi \times \vec{r}_\theta| d\phi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}(\sin\phi\cos\theta)^2 \sin\phi d\phi d\theta = \frac{4}{3}\pi$$

Vector Fields

A vector field on a domain D is a function $\vec{F}$ that assigns to each point (x,y,z) in D a three dimensional vector $\vec{F}(x,y,z)$. In terms of the component functions the vector field $\vec{F}$ is given by

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

One is interested in integrals of the form

$$\iint_S \vec{F} \bullet \vec{n} ds$$

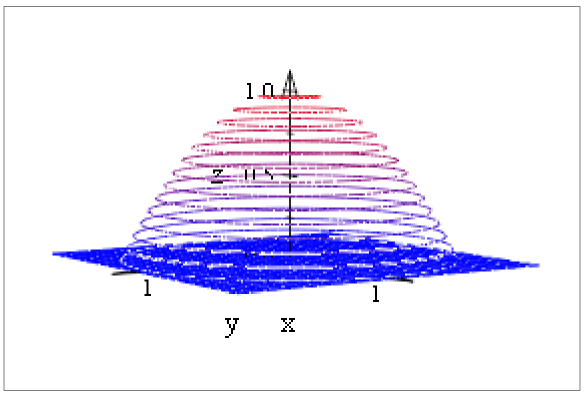
where $\vec{n}$ is a unit normal (perpendicular) vector to this surface S pointing in the outward direction. A unit normal to this surface given in a parametric form is

$$\pm \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

The appropriate sign (either + or –) is chosen that makes the normal point outward. Since $\vec{F} \bullet \vec{n}$ is a scalar we may use our earlier formulation for this surface integral to write

$$\iint_S \vec{F} \bullet \vec{n} ds = \iint_G \vec{F} \bullet \left(\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \right) |\vec{r}_u \times \vec{r}_v| dudv = \iint_G \vec{F} \bullet (\vec{r}_u \times \vec{r}_v) dudv$$

Example:
Evaluate $\iint_S \vec{F} \bullet d\vec{S} = \iint_S \vec{F} \bullet \vec{n} ds$ where $\vec{F} = y\vec{i} + x\vec{j} + z\vec{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.
Solution:
The graph of the surface is shown below.



The surface S consists of the part of the parboiled S_1 between $0 \le z \le 1$ and the circle $x^2 + y^2 \le 1, z = 0,\ S_2$.
On the paraboloid

$$x = u,\ \ y = v,\ \ z = 1 - u^2 - v^2$$

so

$$\vec{r}(u,v) = u\vec{i} + v\vec{j} + (1 - u^2 - v^2)\vec{k}$$

$$\vec{r}_u = \vec{i} - 2u\vec{k}$$

$$\vec{r}_v = j - 2v\vec{k}$$

$$\vec{r}_u \times \vec{r}_v = 2u\vec{i} + 2v\vec{j} + \vec{k}$$

Note that when $x = y = u = v = 0,$ then $\vec{r}_u \times \vec{r}_v = \vec{k}$ which is points in the outward direction. Thus we use $\vec{r}_u \times \vec{r}_v$.

$$\vec{F} \cdot (\vec{r}_u \times \vec{r}_v) = 4uv + (1 - u^2 - v^2)$$

Using polar coordinates $u = r\cos\theta, v = r\sin\theta$ we have

$$\iint_{S_1} \vec{F} \cdot dS = \iint_{S_1} \vec{F} \cdot \vec{n} ds$$
$$= \int\int [4uv + (1 - u^2 - v^2)] dudv$$
$$= \int_0^{2\pi} \int_0^1 (4r^2 \cos\theta \sin\theta + 1 - r^2) r dr d\theta = \frac{\pi}{2}$$

On the unit disk in the x, y –plane centered at the origin we have $\vec{n} = -\vec{k}$, $\vec{F} \cdot \vec{n} = -z = 0$ so

$$\iint_{S_2} \vec{F} \cdot dS = \iint_{S_2} \vec{F} \cdot \vec{n} ds = 0$$

Thus

$$\iint_{S} \vec{F} \cdot dS = \iint_{S} \vec{F} \cdot \vec{n} ds = \frac{\pi}{2}$$

Stokes' Theorem

Let S be a regular surface bounded by a closed curve denoted by ∂S (boundary of S). Let $\vec{F}$ and *curl* $\vec{F}$ be continuous over S. Then

$$\int\int_S curl\vec{F} \cdot \vec{n} ds = \int\int_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n} ds = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

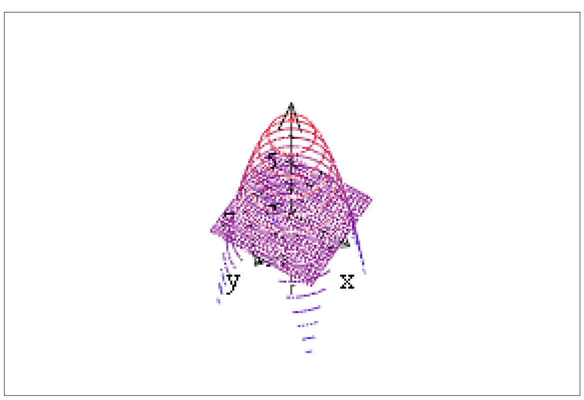
Here the direction of integration around ∂S is positive if the region it encloses is to the left when we go round it with our head in the direction of $\vec{n}$.

Example

Verify that Stokes' Theorem is true for the vector field $\vec{F} = 3y\vec{i} + 4z\vec{j} - 6x\vec{k}$ and S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the x, y –plane, oriented upward.

Solution:

$9 - x^2 - y^2$



For the line integral: The boundary is the circle $z = 0, x^2 + y^2 = 9$. We parametrize the circle as $x = 3\cos\theta, y = 3\sin\theta, \ 0 \leq \theta \leq 2\pi$. Thus

$$\vec{r}(\theta) = 3\cos\theta\vec{i} + 3\sin\theta\vec{j} + 0\vec{k} \quad 0 \le \theta \le 2\pi$$

Then

$$\vec{r}'(\theta) = -3\sin\theta\vec{i} + 3\cos\theta\vec{j}$$

$$\vec{F}(\theta)\cdot\vec{r}'(\theta) = \left(9\sin\theta\vec{i} + 0\vec{j} - 6(3\cos\theta)\vec{k}\right)\cdot\left(-3\sin\theta\vec{i} + 3\cos\theta\vec{j}\right) = -27\sin^2\theta$$

$$\oint_{\partial S}\vec{F}\cdot d\vec{r} = \int_0^{2\pi}(-27\sin^2\theta)d\theta = -27\pi$$

For the surface

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = 9 - r^2,\quad 0 \le \theta \le 2\pi,\ \ 0 \le r \le 3$$

$$\vec{r}(\theta,r) = r\cos\theta\vec{i} + r\sin\theta\vec{j} + (9 - r^2)\vec{k}$$

$$\vec{r}(\theta,r) = (r\cos\theta, r\sin\theta, 9 - r^2)$$

$$\frac{\partial\vec{r}(\theta,r)}{\partial\theta} = (-r\sin\theta, r\cos\theta, 0)$$

$$\frac{\partial\vec{r}(\theta,r)}{\partial r} = (\cos\theta, \sin\theta, -2r)$$

$(\cos\theta,\sin\theta,-2r)\times(-r\sin\theta,r\cos\theta,0) = (2r^2\cos\theta, 2r^2\sin\theta, (\cos^2\theta)r + (\sin^2\theta)r)$

So

$$\vec{r}_r \times \vec{r}_\theta = 2r^2\cos\theta\vec{i} + 2r^2\sin\theta\vec{j} + r\vec{k}$$

Letting $\theta = 0,$ we see that $\vec{r}_u \times \vec{r}_v = 2r^2\vec{i} + r\vec{k},$ which points outward since $r \ge 0.$ Thus we use $\vec{r}_u \times \vec{r}_v.$

$\vec{F} = (3y, 4z, -6x) \quad \nabla\times\vec{F} = (-4, 6, -3)$

$$curl\vec{F}\cdot(\vec{r}_r \times \vec{r}_\theta) = (-4,6,-3)\cdot(2r^2\cos\theta, 2r^2\sin\theta, r)$$

$$= -8r^2\cos\theta + 12r^2\sin\theta - 3r$$

Thus

$$\int\int_S curl\vec{F}\cdot\vec{n}ds = \int_0^{2\pi}\int_0^3(-8r^2\cos\theta + 12r^2\sin\theta - 3r)drd\theta = -27\pi$$

Example: Calculate the work done by the force field

$$\vec{F}(x,y,z) = (x^x + z^2)\vec{i} + (y^y + x^2)\vec{j} + (z^z + y^2)\vec{k}$$

when a particle moves under its influence around the edge of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies in the first octant, in a counterclockwise direction as viewed from above.

Solution:

Given the form of $\vec{F}$ it is clear that evaluating the line integral that gives the work is non-trivial, if not impossible. We shall use Stokes' Theorem to evaluate the line integral and hence find the work.

$\vec{F} = (x^x + z^2, y^y + x^2, z^z + y^2) \quad \nabla\times\vec{F} = (2y, 2z, 2x) = 2y\vec{i} + 2z\vec{j} + 2x\vec{k}$

$$\int\int_S(\vec{\nabla}\times\vec{F})\cdot\vec{n}ds = \oint_{\partial S}\vec{F}\cdot d\vec{r}$$

First we deal with the surface integral by parametrizing the portion of the sphere in the first octant using spherical coordinates. Since the sphere has radius 2, $\rho = 2.$ Thus

$$x = 2\sin\phi\cos\theta,\ \ y = 2\sin\phi\sin\theta,\ \ z = 2\cos\phi \quad 0 \le \phi \le \frac{\pi}{2},\ \ 0 \le \theta \le \frac{\pi}{2}$$

Note that the limits on ϕ and θ come from the fact that we are interested only in the part of the sphere in the first octant.
As usual

$$\vec{r}(\theta,\phi) = 2\sin\phi\cos\theta\vec{i} + 2\sin\phi\sin\theta\vec{j} + 2\cos\phi\vec{k}$$

so

$$\vec{r}_\theta \times \vec{r}_\phi = -4\sin^2\phi\cos\theta\vec{i} - 4\sin^2\phi\sin\theta\vec{j} - 4\sin\phi\cos\phi\vec{k}$$

This normal points "down" (let $\phi = \frac{\pi}{2}, \theta = 0$) so we use $-\vec{r}_\theta \times \vec{r}_\phi$, which points upward.

$$\begin{aligned}\int\int_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}ds &= \int\int_S (\vec{\nabla}\times\vec{F})\cdot(-\vec{r}_\theta\times\vec{r}_\phi)ds \\ &= \int\int_S \left(4\sin\phi\sin\theta\vec{i} + 4\cos\phi\vec{j} + 4\sin\phi\cos\theta\vec{k}\right)\cdot\left(4\sin^2\phi\cos\theta\vec{i} + 4\sin^2\phi\sin\theta\vec{j} + 4\sin\phi\cos\phi\vec{k}\right) \\ &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} (16\sin^3\phi\sin\theta\cos\theta + 16\cos\phi\sin^2\phi\sin\theta + 16\sin^2\phi\cos\phi\cos\theta)d\theta d\phi = 16\end{aligned}$$

The Divergence Theorem

Remark: We shall call a surface S positively oriented if the unit normal $\vec{n}$ is an outer normal; otherwise, S is negatively oriented.

Theorem: Suppose S is a regular, positively oriented, closed surface, and that $\vec{F}$ and $div\,\vec{F}$ are continuous over S and the region V is enclosed by S.

Then

$$\int\int_S \vec{F}\cdot d\vec{S} = \int\int_S \vec{F}\cdot\vec{n}ds = \int\int\int_V div\vec{F}dv = \int\int\int_V \nabla\cdot\vec{F}dv$$

where $\vec{n}$ is the *outward* normal to S.

Example: Verify that the Divergence Theorem is true for the vector field $\vec{F} = 3x\vec{i} + xy\vec{j} + 2xz\vec{k}$ and V is the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.
Solution:
Since $div\vec{F} = 3 + x + 2x = 3 + 3x$, then

$$\int\int\int_V \nabla\cdot\vec{F}dv = \int_0^1\int_0^1\int_0^1 (3x+3)dxdydz = \frac{9}{2}$$

There are six faces to the cube.
Face $x = 0$, then $\vec{n} = -\vec{i}$, so $\vec{F}\cdot\vec{n} = 3x = 0$
Face $x = 1$, then $\vec{n} = \vec{i}$, $\vec{F}\cdot\vec{n} = 3x = 3$, so $\iint_{face\ x=1} \vec{F}\cdot\vec{n}ds = \iint_{face\ x=1} 3ds = 3\times$area of face$= 3$

Face $y = 0, \vec{n} = -\vec{j}$, $\vec{F}\cdot\vec{n} = -xy = 0$

Face $y = 1, \vec{n} = \vec{j}$, $\vec{F}\cdot\vec{n} = xy = x$ so $\int\int_{face\ y=1} \vec{F}\cdot\vec{n}ds = \int_0^1\int_0^1 xdxdz = \frac{1}{2}$

Face $z = 0, \vec{n} = -\vec{k}, \ \vec{F} \cdot \vec{n} = -2xz = 0$

Face $z = 1, \vec{n} = \vec{k}, \ \vec{F} \cdot \vec{n} = 2xz = 2x$ so $\int\int_{face\ z=1} \vec{F} \cdot \vec{n} ds = \int_0^1 \int_0^1 2x dx dy = 1$

Thus

$$\int\int_S \vec{F} \cdot \vec{n} ds = 0 + 3 + 0 + \frac{1}{2} + 0 + 1 = \frac{9}{2}$$

as before.

Example

Use the Divergence Theorem to calculate the surface integral $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F}(x,y,z) = x^3\vec{i} + y^3\vec{j} + z^3\vec{k},$ and S is the sphere $x^2 + y^2 + z^2 = 1$

Solution:

$div\vec{F} = 3(x^2 + y^2 + z^2)$ so

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V 3(x^2 + y^2 + z^2) dV$$

Switching to spherical coordinates we have since

$$dV = \rho^2 \sin\phi d\rho d\theta d\phi$$

and $x^2 + y^2 + z^2 = \rho^2$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V 3(x^2 + y^2 + z^2) dV$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^1 3\rho^4 \sin\phi d\rho d\theta d\phi = \frac{12}{5}\pi$$