Ma 227 Review of Surface Integrals, Stokes’ Theorem, and Divergence Theorem

Surface Integrals
Suppose \( f(x,y,z) \) is a function of three variables whose domain includes a surface \( S \). Then
\[
\iint_S f(x,y,z)\,ds
\]
is called the surface integral of \( f \) over \( S \). Suppose that a surface \( S \) has a vector equation
\[
\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}
\]
where the parameters \( (u,v) \) have values in some domain \( D \). We define the derivatives (tangent vectors)
\[
\vec{r}_u(u,v) = \frac{\partial x(u,v)}{\partial u} \hat{i} + \frac{\partial y(u,v)}{\partial u} \hat{j} + \frac{\partial z(u,v)}{\partial u} \hat{k}
\]
and
\[
\vec{r}_v(u,v) = \frac{\partial x(u,v)}{\partial v} \hat{i} + \frac{\partial y(u,v)}{\partial v} \hat{j} + \frac{\partial z(u,v)}{\partial v} \hat{k}
\]
If \( \vec{r}_u \) and \( \vec{r}_v \) are nonzero and non-parallel in \( D \), the surface integral over \( S \) is given as
\[
\iint_S f(x,y,z)\,ds = \iint_G f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|\,dudv
\]
where \( G \) is the image of the surface \( S \) in the \( u,v \)-plane, and \( f(\vec{r}(u,v)) \) is short for \( f(x(u,v),y(u,v),z(u,v)) \).

Graph of \( z = g(x,y) \)
Any surface with equation
\[
z = g(x,y)
\]
can be regarded as a parametric surface with parametric equations
\[
x = u, \ y = v, \ z = g(u,v)
\]
that is,
\[
\vec{r}(u,v) = \hat{u} + \vec{v} + g(u,v)\hat{k}
\]
Now
\[
\vec{r}_u = \hat{i} + g_u(u,v)\hat{k}
\]
\[
\vec{r}_v = \hat{j} + g_v(u,v)\hat{k}
\]
so that
\[
\vec{r}_u \times \vec{r}_v = -g_u\hat{i} - g_v\hat{j} + \hat{k}
\]
and
\[ |\vec{r}_u \times \vec{r}_v| = [1 + g_u^2 + g_v^2]^{\frac{1}{2}} \]

Because \( u = x, v = y \) we get
\[
\int \int_S f(x,y,z)\,ds = \int \int_G f(x,y,g(x,y))[1 + g_x^2 + g_y^2]^{\frac{1}{2}}\,dxdy
\]

Hence, if we let \( f(x,y,z) = 1, \) we get
\[
\int \int_S f(x,y,z)\,ds = \int \int_G [1 + g_x^2 + g_y^2]^{\frac{1}{2}}\,dxdy
\]

the area of \( S, \) as we should.

In general
\[
\int \int_S f(x,y,z)\,ds = \int \int_G f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|\,dudv
\]
gives for \( f(x,y,z) = 1 \) the area of \( S. \)

In class we evaluated \( \int \int_S f(x,y,z)\,ds \) where \( f = x^2 \) and \( S \) was the part of the cone \( z = x^2 + y^2 \) between the planes \( z = 1 \) and \( z = 2. \) We used spherical coordinates and set \( \rho = \frac{z}{2} \) for the equation of the cone.

Here, let us do a somewhat simplified example.

Example 1: Compute the surface integral \( \int \int_S x^2\,ds \) where \( S \) is the unit sphere \( x^2 + y^2 + z^2 = 1. \)

Solution:
We use spherical coordinates, \( \rho = 1; \ 0 \leq \phi \leq \pi, \ 0 \leq \theta \leq 2\pi \)
\[
x = \sin \phi \cos \theta, \ y = \sin \phi \sin \theta, \ z = \cos \phi
\]
\[
\vec{r}(\phi, \theta) = \sin \phi \cos \theta \vec{i} + \sin \phi \sin \theta \vec{j} + \cos \phi \vec{k}
\]
\[
\vec{r}_\phi = \cos \phi \cos \theta \vec{i} + \cos \phi \sin \theta \vec{j} - \sin \phi \vec{k}
\]
\[
\vec{r}_\theta = -\sin \phi \sin \theta \vec{i} + \sin \phi \cos \theta \vec{j}
\]
so
\[
\vec{r}_\phi \times \vec{r}_\theta = \sin^2 \phi \cos \theta \vec{i} + \sin^2 \phi \sin \theta \vec{j} + \sin \phi \cos \phi \vec{k}
\]
\[
|\vec{r}_\phi \times \vec{r}_\theta| = \sin \phi
\]

Therefore,
\[
\int \int_S x^2\,ds = \int_0^{2\pi} \int_0^\pi (\sin \phi \cos \theta)^2 |\vec{r}_\phi \times \vec{r}_\theta|\,d\phi d\theta
\]
\[
= \int_0^{2\pi} \int_0^\pi (\sin \phi \cos \theta)^2 \sin \phi d\phi d\theta = \frac{4}{3} \pi
\]

**Vector Fields**

A vector field on a domain \( D \) is a function \( \vec{F} \) that assigns to each point \((x,y,z)\) in \( D \) a three dimensional
vector \( \vec{F}(x,y,z) \). In terms of the component functions the vector field \( \vec{F} \) is given by

\[
\vec{F}(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}
\]

One is interested in integrals of the form

\[
\iint_S \vec{F} \cdot \vec{n} \, ds
\]

where \( \vec{n} \) is a unit normal (perpendicular) vector to this surface \( S \) pointing in the outward direction. A unit normal to this surface given in a parametric form is

\[
\vec{n} = \pm \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}
\]

The appropriate sign (either + or −) is chosen that makes the normal point outward. Since \( \vec{F} \cdot \vec{n} \) is a scalar we may use our earlier formulation for this surface integral to write

\[
\iint_S \vec{F} \cdot \vec{n} \, ds = \iint_G \vec{F} \cdot \left( \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \right) |\vec{r}_u \times \vec{r}_v| \, dudv = \iint_G \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, dudv
\]

Example:

Evaluate \( \iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, ds \) where \( \vec{F} = y\hat{i} + x\hat{j} + z\hat{k} \) and \( S \) is the boundary of the solid region \( E \) enclosed by the paraboloid \( z = 1 - x^2 - y^2 \) and the plane \( z = 0 \).

Solution:

The graph of the surface is shown below.

The surface \( S \) consists of the part of the paraboloid \( S_1 \) between \( 0 \leq z \leq 1 \) and the circle \( x^2 + y^2 \leq 1, z = 0, S_2 \).

On the paraboloid

\[
x = u, \quad y = v, \quad z = 1 - u^2 - v^2
\]

so

\[
\vec{r}(u,v) = u\hat{i} + v\hat{j} + (1 - u^2 - v^2)\hat{k}
\]

\[
\vec{r}_u = \hat{i} - 2u\hat{k}
\]

\[
\vec{r}_v = \hat{j} - 2v\hat{k}
\]

\[
\vec{r}_u \times \vec{r}_v = 2u\hat{i} + 2v\hat{j} + \hat{k}
\]

Note that when \( x = y = u = v = 0 \), then \( \vec{r}_u \times \vec{r}_v = \hat{k} \) which is points in the outward direction. Thus we
use $\vec{r}_u \times \vec{r}_v$.
\[
\vec{F} \cdot (\vec{r}_u \times \vec{r}_v) = 4uv + (1 - u^2 - v^2)
\]
Using polar coordinates $u = r \cos \theta, v = r \sin \theta$ we have
\[
\iint_S \vec{F} \cdot dS = \iint_S \vec{F} \cdot \vec{n}ds
= \int \int_{S_1} [4uv + (1 - u^2 - v^2)]dudv
= \int_0^{2\pi} \int_0^1 (4r^2 \cos \theta \sin \theta + 1 - r^2)rdrd\theta = \frac{\pi}{2}
\]
On the unit disk in the $x,y$–plane centered at the origin we have $\vec{n} = -\hat{k}$, $\vec{F} \cdot \vec{n} = -z = 0$ so
\[
\iint_{S_2} \vec{F} \cdot dS = \iint_{S_2} \vec{F} \cdot \vec{n}ds = 0
\]
Thus
\[
\iint_S \vec{F} \cdot dS = \iint_S \vec{F} \cdot \vec{n}ds = \frac{\pi}{2}
\]

**Stokes’ Theorem**

Let $S$ be a regular surface bounded by a closed curve denoted by $\partial S$ (boundary of $S$). Let $\vec{F}$ and $\text{curl} \; \vec{F}$ be continuous over $S$. Then
\[
\iint_S \text{curl} \vec{F} \cdot \vec{n}ds = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}ds = \oint_{\partial S} \vec{F} \cdot d\vec{r}
\]
Here the direction of integration around $\partial S$ is positive if the region it encloses is to the left when we go round it with our head in the direction of $\vec{n}$.

Example

Verify that Stokes’ Theorem is true for the vector field $\vec{F} = 3yi + 4zj − 6xk$ and $S$ is the part of the paraboloid $z = 9 − x^2 − y^2$ that lies above the $x,y$–plane, oriented upward.

Solution:

$9 − x^2 − y^2$

For the line integral: The boundary is the circle $z = 0, x^2 + y^2 = 9$. We parametrize the circle as $x = 3 \cos \theta, y = 3 \sin \theta, \ 0 \leq \theta \leq 2\pi$. Thus
\[ \mathbf{r}(\theta) = 3 \cos \theta \hat{i} + 3 \sin \theta \hat{j} + 0 \hat{k} \quad 0 \leq \theta \leq 2\pi \]

Then
\[ \mathbf{r}'(\theta) = -3 \sin \theta \hat{i} + 3 \cos \theta \hat{j} \]
\[ \mathbf{F}(\theta) \cdot \mathbf{r}'(\theta) = \left(9 \sin \theta \hat{i} + \theta \hat{j} - 6(3 \cos \theta) \hat{k}\right) \cdot \left(-3 \sin \theta \hat{i} + 3 \cos \theta \hat{j}\right) = -27 \sin^2 \theta \]
\[ \oint_{S} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{2\pi} (-27 \sin^2 \theta) d\theta = -27\pi \]

For the surface
\[ x = r \cos \theta, \quad y = r \sin \theta, \quad z = 9 - r^2, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 3 \]
\[ \mathbf{r}(\theta, r) = r \cos \theta \hat{i} + r \sin \theta \hat{j} + (9 - r^2) \hat{k} \]
\[ \mathbf{F}(\theta, r) = (r \cos \theta, r \sin \theta, 9 - r^2) \]
\[ \frac{\partial \mathbf{r}(\theta, r)}{\partial \theta} = (-r \sin \theta, r \cos \theta, 0) \]
\[ \frac{\partial \mathbf{r}(\theta, r)}{\partial r} = (\cos \theta, \sin \theta, -2r) \]
\[ (\cos \theta, \sin \theta, -2r) \times (-r \sin \theta, r \cos \theta, 0) = (2r^2 \cos \theta, 2r^2 \sin \theta, (\cos^2 \theta) r + (\sin^2 \theta) r) \]

So
\[ \mathbf{F}( \mathbf{r}_u \times \mathbf{r}_v ) = 2r^2 \cos \theta \hat{i} + 2r^2 \sin \theta \hat{j} + r \hat{k} \]

Letting \( \theta = 0 \), we see that \( \mathbf{r}_u \times \mathbf{r}_v = 2r^2 \hat{i} + r \hat{k} \), which points outward since \( r \geq 0 \). Thus we use \( \mathbf{r}_u \times \mathbf{r}_v \).
\[ \mathbf{F} = (3y, 4z, -6x) \quad \nabla \times \mathbf{F} = (-4, 6, -3) \]
\[ \text{curl} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v ) = (-4, 6, -3) \cdot (2r^2 \cos \theta, 2r^2 \sin \theta, r) \]
\[ = -8r^2 \cos \theta + 12r^2 \sin \theta - 3r \]

Thus
\[ \int \int_{S} \text{curl} \mathbf{F} \cdot \mathbf{n} dS = \int_{0}^{2\pi} \int_{0}^{3} (-8r^2 \cos \theta + 12r^2 \sin \theta - 3r) dr d\theta = -27\pi \]

Example: Calculate the work done by the force field
\[ \mathbf{F}(x, y, z) = (x^2 + z^2) \hat{i} + (y^2 + x^2) \hat{j} + (z^2 + y^2) \hat{k} \]
when a particle moves under its influence around the edge of the part of the sphere \( x^2 + y^2 + z^2 = 4 \) that lies in the first octant, in a counterclockwise direction as viewed from above.

Solution:
Given the form of \( \mathbf{F} \) it is clear that evaluating the line integral that gives the work is non-trivial, if not impossible. We shall use Stokes’ Theorem to evaluate the line integral and hence find the work.
\[ \mathbf{F} = (x^2 + z^2, y^2 + x^2, z^2 + y^2) \]
\[ \nabla \times \mathbf{F} = (2y, 2z, 2x) = 2yi + 2zj + 2xk \]
\[ \int \int_{S} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \oint_{C} \mathbf{F} \cdot d\mathbf{r} \]

First we deal with the surface integral by parametrizing the portion of the sphere in the first octant using spherical coordinates. Since the sphere has radius 2, \( \rho = 2 \). Thus
\[ x = 2 \sin \phi \cos \theta, \quad y = 2 \sin \phi \sin \theta, \quad z = 2 \cos \phi \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq \frac{\pi}{2} \]
Note that the limits on $\phi$ and $\theta$ come from the fact that we are interested only in the part of the sphere in the first octant.

As usual

$$\vec{r}(\theta, \phi) = 2 \sin \phi \cos \theta \hat{i} + 2 \sin \phi \sin \theta \hat{j} + 2 \cos \phi \hat{k}$$

so

$$\vec{r}_\theta \times \vec{r}_\phi = -4 \sin^2 \phi \cos \theta \hat{i} - 4 \sin^2 \phi \sin \theta \hat{j} - 4 \sin \phi \cos \phi \hat{k}$$

This normal points "down" (let $\phi = \frac{\pi}{2}$, $\theta = 0$) so we use $-\vec{r}_\theta \times \vec{r}_\phi$, which points upward.

$$\int \int_S (\nabla \times \vec{F}) \cdot \vec{n} \, ds = \int \int_S (\nabla \times \vec{F}) \cdot (-\vec{r}_\theta \times \vec{r}_\phi) \, ds$$

$$= \int \int_S \left( 4 \sin \phi \sin \theta \hat{i} + 4 \cos \phi \sin \theta \hat{j} + 4 \sin \phi \cos \phi \hat{k} \right) \cdot \left( 4 \sin^2 \phi \cos \theta \hat{i} + 4 \sin^2 \phi \sin \theta \hat{j} + 4 \sin \phi \cos \phi \hat{k} \right)$$

$$= \int_0^\frac{\pi}{2} \int_0^{\frac{\pi}{2}} \left( 16 \sin^3 \phi \sin \theta \cos \theta + 16 \cos \phi \sin^2 \phi \sin \theta + 16 \sin^2 \phi \cos \phi \cos \theta \right) \, d\theta d\phi = 16$$

**The Divergence Theorem**

Remark: We shall call a surface $S$ positively oriented if the unit normal $\vec{n}$ is an outer normal; otherwise, $S$ is negatively oriented.

Theorem: Suppose $S$ is a regular, positively oriented, closed surface, and that $\vec{F}$ and $\text{div} \vec{F}$ are continuous over $S$ and the region $V$ is enclosed by $S$.

Then

$$\int \int_S \vec{F} \cdot d\vec{S} = \int \int \int_V \text{div} \vec{F} \, dv = \int \int \int_V \nabla \cdot \vec{F} \, dv$$

where $\vec{n}$ is the outward normal to $S$.

Example: Verify that the Divergence Theorem is true for the vector field $\vec{F} = 3x\hat{i} + xy\hat{j} + 2xz\hat{k}$ and $V$ is the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution:

Since $\text{div} \vec{F} = 3 + x + 2x = 3 + 3x$, then

$$\int \int \int_V \nabla \cdot \vec{F} \, dv = \int_0^1 \int_0^1 \int_0^1 (3x + 3) \, dx \, dy \, dz = \frac{9}{2}$$

There are six faces to the cube.

Face $x = 0$, then $\vec{n} = -\hat{i}$, so $\vec{F} \cdot \vec{n} = 3x = 0$

Face $x = 1$, then $\vec{n} = \hat{i}$, $\vec{F} \cdot \vec{n} = 3x = 3$, so $\int \int \int_{face \, x=1} \vec{F} \cdot \vec{n} \, ds = \int \int_0^1 \int_0^1 3 \, dx \, dy \, dz = 3 \times \text{area of face} = 3$

Face $y = 0$, $\vec{n} = -\hat{j}$, $\vec{F} \cdot \vec{n} = -xy = 0$

Face $y = 1$, $\vec{n} = \hat{j}$, $\vec{F} \cdot \vec{n} = xy = x$ so $\int \int \int_{face \, y=1} \vec{F} \cdot \vec{n} \, ds = \int \int_0^1 \int_0^1 x \, dx \, dz = \frac{1}{2}$
Face $z = 0, \vec{n} = -\hat{k}, \vec{F} \cdot \vec{n} = -2xz = 0$

Face $z = 1, \vec{n} = \hat{k}, \vec{F} \cdot \vec{n} = 2xz = 2x$ so $\int \int_{face \ z = 1} \vec{F} \cdot \vec{n} ds = \int_0^1 \int_0^1 2xdxdy = 1$

Thus

$$\int \int_{S} \vec{F} \cdot \vec{n} ds = 0 + 3 + 0 + \frac{1}{2} + 0 + 1 = \frac{9}{2}$$

as before.

Example

Use the Divergence Theorem to calculate the surface integral $\iiint_{S} \vec{F} \cdot d\vec{S}$ where

$\vec{F}(x,y,z) = x^3 \hat{i} + y^3 \hat{j} + z^3 \hat{k}$, and $S$ is the sphere $x^2 + y^2 + z^2 = 1$

Solution:

$div \vec{F} = 3(x^2 + y^2 + z^2)$ so

$$\iiint_{S} \vec{F} \cdot d\vec{S} = \iiint_{V} 3(x^2 + y^2 + z^2) dV$$

Switching to spherical coordinates we have since

$$dV = \rho^2 \sin \phi d\rho d\theta d\phi$$

and $x^2 + y^2 + z^2 = \rho^2$

$$\iiint_{S} \vec{F} \cdot d\vec{S} = \iiint_{V} 3(x^2 + y^2 + z^2) dV = \int_0^\pi \int_0^{2\pi} \int_0^1 3\rho^4 \sin \phi d\rho d\theta d\phi = \frac{12}{5}\pi$$