Ma 227 Review for Systems of DEs

Matrices

Basic Properties

Addition and subtraction:

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$. Then

$$A \pm B = [a_{ij} \pm b_{ij}]_{m\times n}$$

Example:

$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \end{bmatrix}$ $\quad B = \begin{bmatrix} 6 & 4 & 7 \\ -1 & -2 & -6 \end{bmatrix}$

$$A + B = \begin{bmatrix} 1+6 & -2+4 & 3+7 \\ 0-1 & -1-2 & 6-6 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 10 \\ -1 & -3 & 0 \end{bmatrix}$$

Scaler Multiplication:

Let k be a scalar and A a matrix of real numbers of order $m \times n$. Then

$$kA = [k \bullet a_{ij}]_{m\times n}$$

Example:

$$5\begin{bmatrix} -1 & 0 & 5 & 7 \\ 2 & -8 & 4 & 22 \\ -7 & 1 & 0 & 6 \\ 8 & 3 & -3 & 4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 25 & 35 \\ 10 & -40 & 20 & 110 \\ -35 & 5 & 0 & 30 \\ 40 & 15 & -15 & 20 \end{bmatrix}$$

Some Properties of Addition and Scalar Multiplication

Theorem

Let A, B and C be conformable $m \times n$ matrices whose entries are real numbers, and k and p arbitrary scalars. Then

1. $A + B = B + A$.
2. $A + (B + C) = (A + B) + C$
3. There is an $m \times n$ matrix 0 such that $0 + A = A$ for each A.
4. For each A there is an $m \times n$ matrix $-A$ such that $A + (-A) = 0$.
5. $k(A + B) = kA + kB$
6. $(k + p)A = kA + pA$
7. $(kp)A = k(pA)$.

(4) Note that $(-1)A = [-a_{ij}]_{m\times n} \quad \Rightarrow A + (-1)A = 0_{m\times n}$

Remark: We denote $(-1)A$ by $-A$.

The Transpose of a Matrix

If A is an $m \times n$ matrix, the transpose of A, denoted A^T, is the $n \times m$ matrix whose entry a_{st} is the same as the entry a_{ts} in the matrix A. Thus one gets the transpose of A by interchanging the rows and the columns of A.

Example:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -2 \\ 4 & 10 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 10 \\ -1 & -2 & 9 \end{bmatrix}$$

Multiplication:

Definition. Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ be matrices. Then $A\,B$ is the $m \times p$ matrix C, where

$$C = [c_{ij}]_{m\times p} = \left[\sum_{k=1}^{n} a_{ik} b_{kj} \right]_{m\times p}$$

Remark. $A\,B \neq B\,A$ necessarily.
Example:

$$\begin{bmatrix} 1 & -1 & 0 \\ 4 & 1 & -1 \end{bmatrix}_{2\times 3} \times \begin{bmatrix} 3 & 4 \\ -1 & -5 \\ 1 & 2 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} (1)(3) + (-1)(-1) + (0)(1) & (1)(4) + (-1)(-5) + (0)(2) \\ (4)(3) + (1)(-1) + (-1)(1) & (4)(4) + (1)(-5) + (-1)(2) \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} 4 & 9 \\ 10 & 9 \end{bmatrix}_{2\times 2}$$

The following occur often for matrices.

1. $A\,B \neq B\,A$
2. $A\,B = 0$ but neither $A = 0$ or $B = 0$

3. $A\,B = A\,C$ but $B \neq C$

Theorem

Assume that k is an arbitrary scalar, and that $A,\ B,\ C$ and I are matrices of sizes such that the indicated operations can be performed. Then

1. $IA = A, \qquad BI = B$

2. $A(BC) = (AB)C$

3. $A(B+C) = AB + AC, \qquad A(B-C) = AB - AC$

4. $(B+C)A = BA + CA, \qquad (B-C)A = BA - CA$

5. $k(AB) = (kA)B = A(kB)$

6. $(AB)^T = B^T A^T$.

Cramer's Rule

Cramer's Rule: Let A be an $n \times n$ matrix, $A = [a_{ij}]_{n\times n}$ and denote by $A_{(j)}$ the $n \times n$ matrix formed by replacing the elements a_{ij} of the jth column of A by the numbers $k_i,\ i = 1, \ldots\ldots, n$. If $|A| \neq 0$, the system of n linear equations in n unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= k_2 \\ \vdots &= \vdots \\ \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= k_n \end{aligned}$$

has the unique solution

$$x_1 = \frac{\det A_{(1)}}{\det A} \qquad x_2 = \frac{\det A_{(2)}}{\det A}, \ldots x_n = \frac{\det A_{(n)}}{\det A}$$

Example. Solve

$$\begin{aligned} x + 3y - 2z &= 1 \\ 4x - 2y + z &= -15 \\ 3x + 4y - z &= 3 \end{aligned}$$

by Cramer's Rule

$$\det A = \begin{vmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -25$$

$$x = \frac{\begin{vmatrix} 1 & 3 & -2 \\ -15 & -2 & 1 \\ 3 & 4 & -1 \end{vmatrix}}{-25} = -\frac{14}{5}, \quad y = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 4 & -15 & 1 \\ 3 & 3 & -1 \end{vmatrix}}{-25} = \frac{19}{5}, \quad z = \frac{\begin{vmatrix} 1 & 3 & 1 \\ 4 & -1 & -15 \\ 3 & 4 & 3 \end{vmatrix}}{-25} = \frac{19}{5}$$

Systems of Equations: Elimination Using Matrices

Elementary Row Operations On Matrices I

Equivalent Systems

Two linear systems are **equivalent** if they have the same solutions.

Three Elementary Operations

Three basic elementary operations are used to transform systems to equivalent systems. These are:

1. Interchanging the order of the equations in the system.

2. Multiplying any equation by a nonzero constant.

3. Replacing any equation in the system by its sum with a nonzero constant multiple of any other equation in the system (elimination step).

Theorem:

Suppose that an elementary row operation is performed on a system of linear equations. Then the resulting system has the same set of solutions as the original, so the two systems are equivalent.

Operating on the rows of a matrix is equivalent to operating on equations. The row operations that are allowed are the same as the row operations on linear systems of equations:

1. Interchanging the rows.

2. Multiplying any row by a nonzero constant.

3. Replacing any row by its sum with a nonzero constant multiple of any other row. (Add a multiple of one row to a different row.)

Gaussian Elimination

Definition: A matrix is said to be in row-echelon form (and will be called a row-echelon matrix) if it satisfies the following three conditions:

1. All zero rows (consisting entirely of zeroes) are at the bottom.

2. The first nonzero entry from the left in each nonzero row is a $1,$ called the leading 1 for that row.

3. Each leading 1 is to the right of all leading $1's$ in the rows above it.

Definition: A row-echelon matrix is said to be in reduced row-echelon form (and will be called a reduced row-echelon matrix) if it satisfies the following condition:

4. Each leading 1 is the only nonzero entry in its column.

Example:

Reduce the matrix

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -3 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & -7 & 8 \end{bmatrix}$$

to row-reduced echelon form.

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -3 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & -7 & 8 \end{bmatrix} \to_{(2)R_1+R_3;\ (2)R_1+R_4} \begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & -1 & 0 & 4 & -8 \\ 0 & 0 & 1 & -3 & 0 \end{bmatrix}$$

$$\to_{(-1)R_1;\ (-1)R_3} \begin{bmatrix} 1 & 1 & 0 & -2 & 4 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 1 & 0 & -4 & 8 \\ 0 & 0 & 1 & -3 & 0 \end{bmatrix}$$

$$\to_{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 & -2 & 4 \\ 0 & 1 & 0 & -4 & 8 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{bmatrix} \to_{(-1)R_3+R_4} \begin{bmatrix} 1 & 1 & 0 & -2 & 4 \\ 0 & 1 & 0 & -4 & 8 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to_{(-1)R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & 2 & -4 \\ 0 & 1 & 0 & -4 & 8 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example:

Solve the system $AX = C$, where

$$A = \begin{bmatrix} -1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & -7 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } C = \begin{bmatrix} -4 \\ 0 \\ 0 \\ 8 \end{bmatrix}$$

From the previous example

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -3 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & -7 & 8 \end{bmatrix}, \text{ row echelon form: } \begin{bmatrix} 1 & 0 & 0 & 2 & -4 \\ 0 & 1 & 0 & -4 & 8 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the solution is: $\{x_3 = 3x_4, x_1 = -2x_4 - 4, x_2 = 4x_4 + 8, x_4 = x_4\}$

Inverse of a Matrix

Definition: If A is a square $n \times n$ matrix, a matrix A^{-1} is called the inverse of A if and only if

$$AA^{-1} = I = A^{-1}A$$

A matrix A that has an inverse is called an invertible or nonsingular matrix.

Example:

Find A^{-1} for $A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$. We form $\left[\begin{array}{ccc|ccc} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$

$$\begin{bmatrix} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow_{(-2)R_2+R_1;\ (-1)R_2+R_3} \begin{bmatrix} 0 & -1 & 3 & 1 & -2 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{bmatrix}$$

$$\rightarrow_{(-1)R_1+R_3;} \begin{bmatrix} 0 & -1 & 3 & 1 & -2 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{bmatrix}$$

$$\rightarrow_{(4)R_1+R_2;\ \left(-\frac{1}{2}\right)R_3} \begin{bmatrix} 0 & -1 & 3 & 1 & -2 & 0 \\ 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \rightarrow_{(-3)R_3+R_1;\ (-11)R_3+R_2} \begin{bmatrix} 0 & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ 1 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\rightarrow_{(-1)R_1;\ \ R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

Eigenvalues

Definition: The values of λ such that $\det(A-\lambda I) = 0$ are called eigenvalues. The vector X corresponding to an eigenvalue is called an eigenvector of the matrix A.

Example Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

.

Solution:

$$\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix}$$
$$= -2+\lambda+2\lambda^2-\lambda^3$$
$$= -\lambda^2(\lambda-2)+(\lambda-2)$$
$$= (1-\lambda^2)(\lambda-2)$$

Thus $\det(A-\lambda I) = 0 \Rightarrow$eigenvalues $\lambda = -1, 1, 2.$

$(A-\lambda I)X = 0 \Rightarrow$

$$(1-\lambda)x_1 + x_2 - 2x_3 = 0$$
$$-x_1 + (2-\lambda)x_2 + x_3 = 0$$
$$0x_1 + x_2 + (-1-\lambda)x_3 = 0$$

$\lambda = -1$

$$2x_1 + x_2 - 2x_3 = 0$$
$$-x_1 + 3x_2 + x_3 = 0$$
$$0x_1 + x_2 + 0x_3 = 0$$

Thus $x_2 = 0, x_1 = x_3$ or $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \leftrightarrow -1.$ Similarly,

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \text{ eigenvectors: } \left\{\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\right\} \leftrightarrow 2, \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\} \leftrightarrow -1, \left\{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right\}$$

Example **Repeated Eigenvalues** Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & -4 & 4 \\ 3 & -2 & 2 \end{bmatrix}$$

Solution:

$$\det(A - rI) = \begin{vmatrix} 1-r & -2 & 4 \\ 3 & -4-r & 4 \\ 3 & -2 & 2-r \end{vmatrix} = -r^3 - r^2 + 8r + 12 = -(r-3)(r+2)^2$$

Thus the eigenvalues are 3 and –2 and –2 is a repeated eigenvalue with multiplicity two. The system of equations $(A - rI)X = 0$ is, for this matrix,

$$\begin{aligned} (1-r)x_1 - 2x_2 + 4x_3 &= 0 \\ 3x_1 - (4+r)x_2 + 4x_3 &= 0 \\ 3x_1 - 2x_2 + (2-r)x_3 &= 0 \end{aligned}$$

Setting $r = 3$ yields

$$\begin{aligned} -2x_1 - 2x_2 + 4x_3 &= 0 \\ 3x_1 - 7x_2 + 4x_3 &= 0 \\ 3x_1 - 2x_2 - x_3 &= 0 \end{aligned}$$

The augmented matrix for this system is $\begin{bmatrix} -2 & -2 & 4 & 0 \\ 3 & -7 & 4 & 0 \\ 3 & -2 & -1 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so the solutions of the above system are also the solutions of the system

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

Thus $x_1 = x_2 = x_3$ and an eigenvector corresponding to $r = 3$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Setting $r = -2$ in the system $(A - rI)X = 0$ yields

$$\begin{aligned} 3x_1 - 2x_2 + 4x_3 &= 0 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \end{aligned}$$

The augmented matrix for this system is $\begin{bmatrix} 3 & -2 & 4 & 0 \\ 3 & -2 & 4 & 0 \\ 3 & -2 & 4 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & -\frac{2}{3} & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus, we have the one equation

$$x_1 - \frac{2}{3}x_2 + \frac{4}{3}x_3 = 0$$

To get two linearly independent vectors we first take $x_3 = 0$ and get $x_1 = \frac{2}{3}x_2$. Letting $x_2 = 1$ yields the eigenvector $\begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$.

To get a second vector we set $x_2 = 0$ and get $x_1 = -\frac{4}{3}x_3$. Letting $x_3 = 1$ yields the eigenvector $\begin{bmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$.

Example Complex Eigenvalues Find the eigenvalues and eigenvectors of the matrix A.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Solution. We note the following.

If $r_1 = \alpha + i\beta$ is a solution of the equation that determines the eigenvalues, namely,

$$p(r) = \det(A - rI) = 0$$

then $r_2 = \alpha - i\beta$ is also a solution of this equation, and hence is an eigenvalue. Recall that r_2 is called the complex conjugate of r_1 and $\overline{r}_1 = r_2$.

Let $\mathbf{z} = \mathbf{a} + i\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are real vectors, be an eigenvector corresponding to r_1. Then it is not hard to see that $\overline{\mathbf{z}} = \mathbf{a} - i\mathbf{b}$ is an eigenvector corresponding to r_2. Since

$$A\mathbf{z} = r_1\mathbf{z} = r_1 I\mathbf{z}$$

then

$$(A - r_1 I)\mathbf{z} = 0$$

Taking the conjugate of this equation and noting that since A and I are real matrices then $\overline{A} = A$ and $\overline{I} = I$

$$\overline{(A - r_1 I)\mathbf{z}} = (A - \overline{r}_1 I)\overline{\mathbf{z}} = (A - r_2 I)\overline{\mathbf{z}} = 0$$

so $\overline{\mathbf{z}}$ is an eigenvector corresponding to r_2.

We find the eigenvalues for matrix A first.

$$\det(A - rI) = \begin{vmatrix} 2-r & -1 & 0 \\ 2 & 1-r & 1 \\ 0 & 2 & 1-r \end{vmatrix} \begin{matrix} 2-r & -1 \\ 2 & 1-r \\ 0 & 2 \end{matrix}$$

$$= (2-r)(1-r)^2 - 2(1)(2-r) + 2(1-r)$$

$$= (2-r)[(1 - 2r + r^2) - 2] + 2 - 2r$$

$$= (2-r)(-1 - 2r + r^2) + 2 - 2r$$

$$= -2 + r - 4r + 2r^2 + 2r^2 - r^3 + 2 - 2r$$

$$= -5r + 4r^2 - r^3 = -r(r^2 - 4r + 5)$$

Clearly one root is $r = 0$. Using the quadratic formula, the others are

$$r = \frac{4 \pm \sqrt{4^2 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2}$$

$$= 2 \pm i$$

The system of equations for the eigenvectors is

$$(2-r)x_1 - x_2 = 0$$

$$2x_1 + (1-r)x_2 + x_3 = 0$$

$$2x_2 + (1-r)x_3 = 0$$

For $r = 0$, we solve

$$(A - 0I)X = 0$$

Using elimination on the augmented matrix, we have

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -.5 & 0 & 0 \\ 0 & 1 & .5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & .25 & 0 \\ 0 & 1 & .5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $x_1 = -\frac{1}{4}x_3$ and $x_2 = -\frac{1}{2}x_3$ where x_3 is arbitrary. Letting $x_3 = 4$ we have that the eigenvector is any multiple of

$$\begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$$

Similarly, for $r = 2 + i$, we have the following. [The first step is an extra step of multiplying the first row by $2i$ to show how this goes.]

$$\begin{bmatrix} -i & -1 & 0 & 0 \\ 2 & -1-i & 1 & 0 \\ 0 & 2 & -1-i & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -2i & 0 & 0 \\ 2 & -1-i & 1 & 0 \\ 0 & 2 & -1-i & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -2i & 0 & 0 \\ 0 & -1+i & 1 & 0 \\ 0 & 2 & -1-i & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 2 & -2i & 0 & 0 \\ 0 & 2 & -1-i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 1-i & 0 \\ 0 & 2 & -1-i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $2x_1 = (-1+i)x_3$ and $2x_2 = (1+i)x_3$. Again, the third component is arbitrary and any multiple of

$$\begin{bmatrix} -1+i \\ 1+i \\ 2 \end{bmatrix}$$

is an eigenvector.

Finally, since the entries in the matrix are all real, both eigenvalues and eigenvectors come in complex conjugate pairs and for $r = 2 - i$, eigenvectors are multiples of

$$\begin{bmatrix} -1-i \\ 1-i \\ 2 \end{bmatrix}.$$

Matrix Methods for Linear Systems of Differential Equations

Linear Systems in Normal Form

A system of n linear differential equations is in normal form if it is expressed as

$$x'(t) = A(t)x(t) + f(t)$$

where $x(t)$ and $f(t)$ are $n \times 1$ column vectors and $A(t) = [a_{ij}(t)]_{n\times n}$.
A system is called homogeneous if $f(t) = 0;$ otherwise it is called nonhomogeneous. When the elements of A are constants, the system is said to have constant coefficients.

Example Express the equation

$$y''' - 6y'' + 11y' - 6ty = \cos t$$

in normal form

$$x'(t) = A(t)x(t) + f(t)$$

Solution: Defining

$$x_1 = y, x_2 = y', x_3 = y''$$

we have

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 6tx_1 - 11x_2 + 6x_3 + \cos t \end{aligned}$$

Thus

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6t & -11 & 6 \end{bmatrix} \text{ and } f(t) = \begin{bmatrix} 0 \\ 0 \\ \cos t \end{bmatrix}$$

Solving Normal Systems

1. To determine a general solution to the $n \times n$ homogeneous system $x' = Ax$:

- **a.** Find a fundamental solution set $\{x_1, \ldots\ldots, x_n\}$ that consists of n linearly independent solutions to the homogeneous equation.
- **b.** Form the linear combination

$$x = Xc = c_1x_1 + \cdots + c_nx_n$$

where $c = col(c_1, \ldots\ldots, c_n)$ is any constant vector and $X = [x_1, \ldots\ldots, x_n]$ is the fundamental matrix, to obtain a general solution.

Theorem

Suppose the $n \times n$ constant matrix A has n linearly independent eigenvectors $u_1, u_2, \ldots, u_n$. Let r_i be the eigenvalue corresponding to the u_i. Then

$$\{e^{r_1t}u_1, e^{r_2t}u_2, \ldots\ldots, e^{r_nt}u_n\}$$

is a fundamental solution set on $(-\infty,\infty)$ for the homogeneous system $x' = Ax$. Hence the general solution of $x' = Ax$ is

$$x(t) = c_1 e^{r_1 t} u_1 + \cdots + c_n e^{r_n t} u_n$$

where $c_1,\ldots,c_n$ are arbitrary constants.

Example Find a general solution of

$$x' = \begin{bmatrix} 5 & 4 \\ -1 & 0 \end{bmatrix} x$$

$\begin{bmatrix} 5 & 4 \\ -1 & 0 \end{bmatrix}$, eigenvectors: $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} -4 \\ 1 \end{bmatrix}\right\} \leftrightarrow 4$

Thus $x(t) = c_1 e^{t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

Thus the solution is

$$\begin{aligned} x_1(t) &= -c_1 e^{t} - 4c_2 e^{4t} \\ x_2(t) &= c_1 e^{t} + c_2 e^{4t} \end{aligned}$$

Example Find a fundamental matrix for the system

$$x'(t) = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix} x(t)$$

Solution:

$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$, eigenvectors: $\left\{\begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \end{bmatrix}\right\} \leftrightarrow -1, \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\} \leftrightarrow 2, \left\{\begin{bmatrix} -1 \\ 1 \\ 2 \\ 8 \end{bmatrix}\right\} \leftrightarrow 7,$
$\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\} \leftrightarrow 3$

Hence the four linearly independent solutions are

$$e^{-t}\begin{bmatrix}1\\-3\\0\\0\end{bmatrix}, e^{2t}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, e^{7t}\begin{bmatrix}-1\\1\\2\\8\end{bmatrix}, e^{3t}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

Therefore a fundamental matrix is

$$\begin{bmatrix} e^{-t} & e^{2t} & -e^{7t} & e^{3t} \\ -3e^{-t} & 0 & e^{7t} & 0 \\ 0 & 0 & 2e^{7t} & e^{3t} \\ 0 & 0 & 8e^{7t} & 0 \end{bmatrix}$$

Example Solve the initial value problem

$$x'(t) = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix} x(t)$$

$$x(0) = \begin{bmatrix}1\\-1\\1\\0\end{bmatrix}$$

We know from above that the solution general solution to the system is

$$x(t) = c_1 e^{-t}\begin{bmatrix}1\\-3\\0\\0\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + c_3 e^{7t}\begin{bmatrix}-1\\1\\2\\8\end{bmatrix} + c_4 e^{3t}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

$$x(t) = \begin{bmatrix} c_1 e^{-t} + c_2 e^{2t} - c_3 e^{7t} + c_4 e^{3t} \\ -3c_1 e^{-t} + c_3 e^{7t} \\ 2c_3 e^{7t} + c_4 e^{3t} \\ 8c_3 e^{7t} \end{bmatrix}$$

Then

$$x(0) = \begin{bmatrix} c_1 + c_2 - c_3 + c_4 \\ -3c_1 + c_3 \\ 2c_3 + c_4 \\ 8c_3 \end{bmatrix} = \begin{bmatrix}1\\-1\\1\\0\end{bmatrix}$$

. Therefore we must solve the system

$$\begin{aligned} c_1 + c_2 - c_3 + c_4 &= 1 \\ -3c_1 + c_3 &= -1 \\ 2c_3 + c_4 &= 1 \\ 8c_3 &= 0 \end{aligned}$$, Solution is: $\left\{c_3 = 0, c_4 = 1, c_1 = \frac{1}{3}, c_2 = -\frac{1}{3}\right\}$,and $x(t) = \begin{bmatrix} \frac{1}{3}e^{-t} - \frac{1}{3}e^{2t} + e^{3t} \\ -e^{-t} \\ e^{3t} \\ 0 \end{bmatrix}$

Complex Eigenvalues

Consider

$$x'(t) = Ax(t) \qquad (*)$$

in the case where A is a real matrix and the eigenvalues are complex. Denoting the eigenvalues by $\alpha \pm i\beta$, let $\mathbf{z} = \mathbf{a} + i\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are real vectors, be an eigenvector corresponding to the eigenvector r_1. Then

$$\mathbf{x}_1(t) = e^{\alpha t}(\cos\beta t\mathbf{a} - \sin\beta t\mathbf{b})$$
$$\mathbf{x}_2(t) = e^{\alpha t}(\sin\beta t\mathbf{a} + \cos\beta t\mathbf{b})$$

are two real linearly independent solutions of the system $(*)$.

Example Find the general solution of

$$x'(t) = \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix} x(t)$$

Solution: This is problem 1 on page 573 of our DEs text and was assigned for homework.
Eigenvalues:

$$\det(A - rI) = \begin{vmatrix} 2-r & -4 \\ 2 & -2-r \end{vmatrix} = r^2 + 4 = 0 \Rightarrow r = \pm 2i = \alpha \pm i\beta, so \alpha = 0, \beta = 2$$

Eigenvectors:
$r = 2i$:

$$\begin{bmatrix} 2-2i & -4 \\ 2 & -2-2i \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2-2i & -4 & 0 \\ 2 & -2-2i & 0 \end{bmatrix}$$

R_1 says $(2-2i)u_1 = 4u_2 \Rightarrow u_2 = \left(\frac{2-2i}{4}\right)u_1 = \left(\frac{1}{2} - \frac{i}{2}\right)u_1$. Let $u_1 = s$;

then $\vec{u} = \begin{bmatrix} s \\ \left(\frac{1}{2} - \frac{i}{2}\right)s \end{bmatrix} = s\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} + is\begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}$. Let $s = 2$:

$\Rightarrow$

$$\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \vec{a} + i\vec{b}.$$

So the general solution is

$$\begin{aligned} \vec{x}(t) &= c_1\left\{e^{0t}\cos 2t\begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{0t}\sin 2t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right\} + c_2\left\{e^{0t}\sin 2t\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{0t}\cos 2t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right\} \\ &= c_1\begin{bmatrix} 2\cos 2t \\ \cos 2t + \sin t2t \end{bmatrix} + c_2\begin{bmatrix} 2\sin 2t \\ \sin 2t - \cos 2t \end{bmatrix} \end{aligned}$$

Nonhomogeneous Systems

Undetermined Coefficients

Consider the nonhomogeneous constant coefficient system

$$x'(t) = Ax(t) + f(t)$$

Example Find the general solution of

$$x'(t) = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 2e^t \\ 4e^t \\ -2e^t \end{bmatrix}$$

Solution:
We first find the homogeneous solution.

$$\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \text{ eigenvectors: } \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\} \leftrightarrow -3, \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \leftrightarrow 3$$

Since these eigenvectors are linearly independent, then

$$x_h(t) = c_1 e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

We seek a particular solution of the form

$$x_p(t) = e^t \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Then

$$x_p'(t) = e^t \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = Ax_p(t) + \begin{bmatrix} 2e^t \\ 4e^t \\ -2e^t \end{bmatrix} = e^t \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} 2e^t \\ 4e^t \\ -2e^t \end{bmatrix}$$

$$= e^t \left(\begin{bmatrix} a_1 - 2a_2 + 2a_3 \\ -2a_1 + a_2 + 2a_3 \\ 2a_1 + 2a_2 + a_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \right)$$

Thus

$$a_1 = a_1 - 2a_2 + 2a_3 + 2$$
$$a_2 = -2a_1 + a_2 + 2a_3 + 4$$
$$a_3 = 2a_1 + 2a_2 + a_3 - 2$$

Or

$$2a_2 - 2a_3 = 2$$
$$2a_1 - 2a_3 = 4$$
$$2a_1 + 2a_2 = 2$$

Solution is: $\{a_2 = 0, a_1 = 1, a_3 = -1\}$

Therefore

$$x_p(t) = e^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and

$$x(t) = x_h(t) + x_p(t) = c_1 e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Example a) Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

Solution: We solve $\det(A - rI) = 0$.

$$\det(A - rI) = \begin{vmatrix} 2-r & -1 \\ 1 & 2-r \end{vmatrix}$$

$$= (2-r)^2 + 1$$

$$(2-r)^2 = -1$$

$$2 - r = \pm i$$

$$r = 2 \pm i$$

So, the eigenvalues are a complex conjugate pair. We find the eigenvector for one and take the complex conjugate to get the other. For $r = 2 + i,$ we solve

$$(A - rI)u = 0$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus we have the equations

$$-iu_1 - u_2 = 0$$

$$u_1 - iu_2 = 0$$

The second row is redundant, so $-iu_1 - u_2 = 0$ or $u_2 = -i \cdot u_1$. Hence any multiple of $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector for $r = 2 + i$. Then an eigenvector corresponding to $r = 2 - i$ is $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

b) Find the [real] general solution to

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 12e^{2t} \end{bmatrix}.$$

Solution: The solution is the general solution (x_h) to the homogeneous equation plus one [particular] solution (x_p) to the full non-homogeneous equation. First we'll find x_p. It is in the form

$$x_p = \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{2t} \end{bmatrix}$$

Substituting into the D.E., we obtain

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2c_1e^{2t} \\ 2c_2e^{2t} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1e^{2t} \\ c_2e^{2t} \end{bmatrix} + \begin{bmatrix} 0 \\ 12e^{2t} \end{bmatrix}$$

Hence

$$\begin{bmatrix} 2c_1e^{2t} \\ 2c_2e^{2t} \end{bmatrix} = \begin{bmatrix} 2c_1e^{2t} - c_2e^{2t} \\ c_1e^{2t} + 2c_2e^{2t} \end{bmatrix} + \begin{bmatrix} 0 \\ 12e^{2t} \end{bmatrix} = \begin{bmatrix} 2c_1e^{2t} - c_2e^{2t} \\ c_1e^{2t} + 2c_2e^{2t} + 12e^{2t} \end{bmatrix}$$

We can divide by e^{2t} (which is never zero) and move the unknowns to the left side to obtain

$$\begin{bmatrix} -c_2 \\ -c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \end{bmatrix}$$

$$x_p = \begin{bmatrix} -12e^{2t} \\ 0 \end{bmatrix}$$

b) Find the [real] general solution to

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 12e^{2t} \end{bmatrix}.$$

To find a solution to the homogeneous solution we use the eigenvalue $2 + i = \alpha + i\beta$ and the corresponding eigenvector $\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \mathbf{a} + i\mathbf{b}$.

Since

$$\mathbf{x}_1(t) = e^{\alpha t}(\cos\beta t\mathbf{a} - \sin\beta t\mathbf{b})$$
$$\mathbf{x}_2(t) = e^{\alpha t}(\sin\beta t\mathbf{a} + \cos\beta t\mathbf{b})$$

then

$$\mathbf{x}_1(t) = e^{2t}\left(\cos t\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$
$$\mathbf{x}_2(t) = e^{2t}\left(\sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

Hence

$$\begin{aligned} \mathbf{x}_h(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) \\ &= c_1\begin{bmatrix} e^{2t}\cos t \\ e^{2t}\sin t \end{bmatrix} + c_2\begin{bmatrix} e^{2t}\sin t \\ -e^{2t}\cos t \end{bmatrix} \end{aligned}$$

Or, for the solution to the homogeneous equation, we may use one of the eigenvalues and eigenvectors found in 2a to write a complex solution and break it into real and imaginary parts. We'll use $2 + i$.

$$\begin{aligned} x &= e^{(2+i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{2t}(\cos t + i\sin t)\begin{bmatrix} 1 \\ -i \end{bmatrix} \\ &= \begin{bmatrix} e^{2t}\cos t + ie^{2t}\sin t \\ e^{2t}\sin t - ie^{2t}\cos t \end{bmatrix} \\ x_h &= c_1\begin{bmatrix} e^{2t}\cos t \\ e^{2t}\sin t \end{bmatrix} + c_2\begin{bmatrix} e^{2t}\sin t \\ -e^{2t}\cos t \end{bmatrix} \\ &= \begin{bmatrix} e^{2t}\cos t & e^{2t}\sin t \\ e^{2t}\sin t & -e^{2t}\cos t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{aligned}$$

Finally, we add to obtain the desired solution.

$$x = \begin{bmatrix} e^{2t}\cos t & e^{2t}\sin t \\ e^{2t}\sin t & -e^{2t}\cos t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -12e^{2t} \\ 0 \end{bmatrix}$$