Ma 529
Lecture I

1. Foundations of Math Analysis

Limits and Continuity

Consider the the function f defined by

$$f(x) = \frac{x^2-3x+2}{x-2} \qquad x \neq 2.$$

The domain of

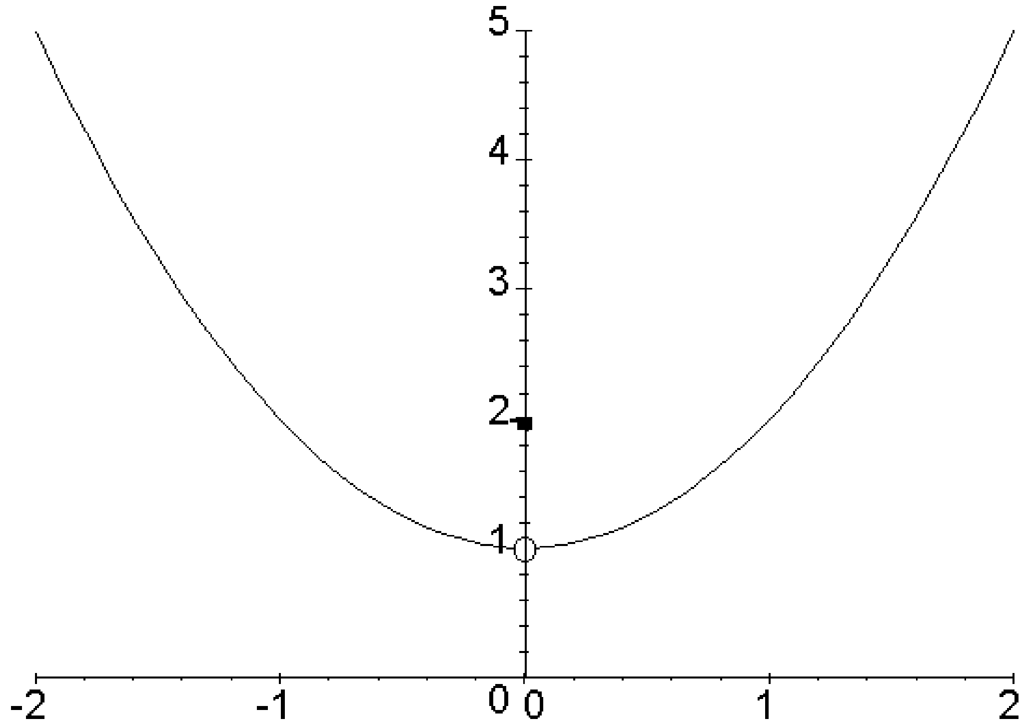
$f(x) = \{x \mid x \neq 2, x\ real\}$. Note that $f(x) = \frac{(x-2)(x-1)}{x-2} = x - 1$ if $x \neq 2$. Thus

$\lim\limits_{x \to 2} \frac{x^2-3x+2}{x-2} = 1$. Note that $\lim\limits_{x \to 2} f(x) = 1$ although $f(2)$ is not defined.

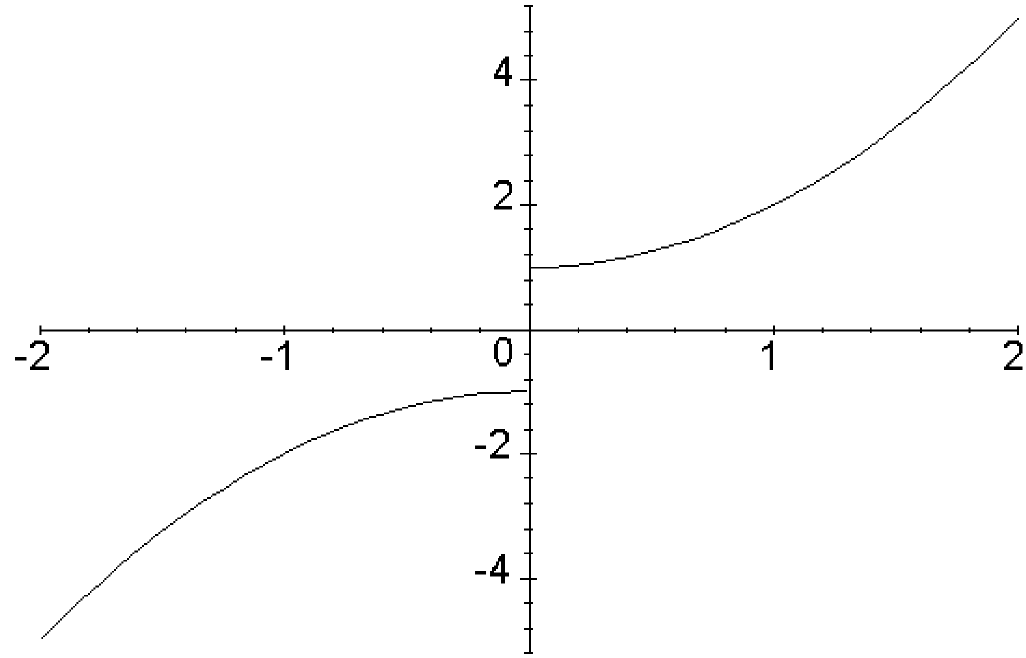
Example. Even if $f(a)$ exists it is not necessarily true that $\lim\limits_{x \to a} f(x) = f(a)$.

Consider the two functions:

$$f(x) = \begin{cases} x^2+1 & \text{if } |x|>0 \\ 2 & \text{if } x=0 \end{cases}$$



$$g(x) = \begin{cases} x^2+1 & \text{if } x \geq 0 \\ -x^2-1 & \text{if } x < 0 \end{cases}$$



Note that domain of $f(x) =$ domain of $g(x) = R$. However $\lim_{x \to 0} f(x) = 1 \neq f(0)$ whereas $\lim_{x \to 0} g(x)$ does not exist.

Definition: Let f be a real-valued function of a real variable. Then the limit as x approaches a of $f(x)$ is b, is written $\lim_{x \to a} f(x) = b$ if for any $\in > 0$, $\exists$ a $\delta > 0$ such that wherever x is in the domain of f and $0 < |x - a| < \delta$, then $|f(x) - b| < \in$.

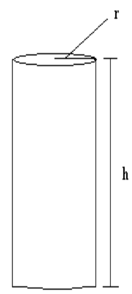
The special case in which $\lim_{x \to a} f(x) = f(a)$ is important. This is the case for a function f defined $\forall \times \in R$ whose graph has no breaks. Such a function is called continuous. To be precise:

Definition: A real-valued function of a real variable is continuous at a if a is in domain of f and $\lim_{x \to a} f(x) = f(a)$. The function f is simply said to be continuous if it is continuous at every number in its domain.

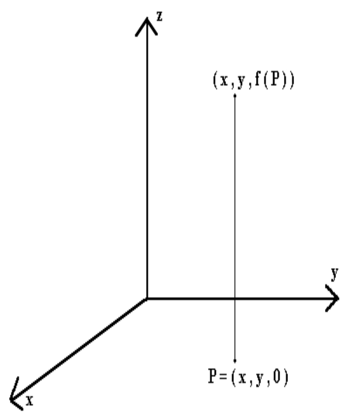
Partial Differentiation

Functions of Two Variables:

The volume of a right circular cylinder of radius r and altitude h is $V = \pi r^2 h$. Clearly V changes as r and h change, i.e., V is a function of the 2 variables r and h.



More generally, if z is uniquely determined by values of x and y, then we say z is a function of x and y and write $z = f(x, y)$. Another way of saying this is as follows: Consider a 3-dimensional coordinate system x, y, z. Then if we consider some point $P = (x, y) \Rightarrow z = f(P)$ and (x, y, z) is a point in 3 space.



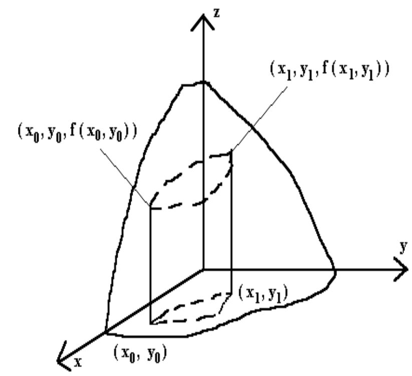
Example. $z = \sqrt{r^2 - x^2 - y^2}$ is the upper half of a sphere of radius r centered at the origin.

Partial Derivatives

In general the graph of $z = f(x,y)$ is a <u>surface</u> in x, y, z-space.

We desire to talk about the derivative of $z = f(x,y)$. Now if we have a function $g(x)$ of one variable, then we know that $g'(x_0)$ is the rate of change of the graph of $y = g(x)$ at $(x_0, g(x_0))$.

Question: Given a point $P_0 = (x_0, y_0)$, what is the rate of change of $z = f(x,y)$ at $(x_0, y_0, f(x_0, y_0))$?



It is clear from the picture that the rate of change depends upon the direction that P_0 is approached from. That is, if $P_1 = (x_1, y_1)$ is another point in the x, y-plane, the rate of change depends upon which curve in the x, y-plane we move along to get to P_0 .

We shall restrict our attention to approaching P_0 from 2 directions, namely along a line parallel to the x-axis or along a line parallel to the y-axis.

Consider approach to the point P_0 along a line $y = y_0$, i.e. parallel to the x-axis. Now $\triangle z = f(x_1, y_0) - f(x_0, y_0)$

$\triangle x = x_1 - x_0$

$\Rightarrow \frac{\triangle z}{\triangle x} = \frac{f(x_1, y_0) - f(x_0, y_0)}{\triangle x}$

or $\frac{\triangle z}{\triangle x} = \frac{f(x_0 + \triangle x, y_0) - f(x_0, y_0)}{\triangle x}$

$\Rightarrow$ rate of change of f along the line $y = y_0$ at (x_0, y_0) is

$$\lim_{\triangle x \to 0} \frac{f(x_0 + \triangle x, y_0) - f(x_0, y_0)}{\triangle x} .$$

Definition. **The first partial derivative of $f(x,y)$ with respect to x at a point (x,y), denoted by $\frac{\partial f}{\partial x}$ or f_x , is**

$$\frac{\partial f}{\partial x} = \lim_{\triangle x \to 0} \frac{f(x+\triangle x,y)-f(x,y)}{\triangle x}$$

Similarly, the rate of change of f along $x = x_0$ at (x_0 , y_0) is called the first partial derivative of $f(x,y)$ with respect to y at (x_0 , y_0). At any point (x,y) we have

$$\frac{\partial f}{\partial y} = f_y = \lim_{\triangle y \to 0} \frac{f(x, y+\triangle y)-f(x,y)}{\triangle y} .$$

The actual computing of a partial derivative is straight-forward. For example, to get $\frac{\partial f}{\partial x}$ the above definition says hold y fixed and differentiate with respect to x.

Example. $z = f(x,y) = 100 - x^2 + y^2$

$\frac{\partial f}{\partial x} = -2x$ $\qquad$ $\frac{\partial f}{\partial y} = 2y$

Example. $f(x,y) = cosh\left(\frac{y}{x}\right)$ $\qquad$ **(Note:** $cosh\ t = \frac{e^t+e^{-t}}{2}$ **and** $sinh\ t = \frac{e^t-e^{-t}}{2}$**.)**

$\frac{\partial f}{\partial y} = \frac{1}{x}\, sinh\left(\frac{y}{x}\right)$ $\qquad$ $\frac{\partial f}{\partial x} = \left(-\frac{y}{x^2}\right) sinh\left(\frac{y}{x}\right)$.

Higher Partial Derivatives

If we have $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, then we may take their partials with respect to x or y. Thus

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$ $\qquad$ $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$ $\qquad$ $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$

Example. $f(x,y) = x^3y^4 - 2x^2e^y$ $\quad$ $f_x = 3x^2y^4 - 4xe^y$ $\quad$ $f_y = 4x^3y^3 - 2x^2e^y$
$f_{xx} = 6xy^4 - 4e^y$ $\qquad$ $f_{yy} = 12x^3y^2 - 2x^2e^y$

Example: $f = x^y$
$\frac{\partial f}{\partial y} = x^y\ ln\ x$ **(1)Recall** $\frac{d}{dx}\, a^u = a^u\ ln\ a\ \frac{du}{dx}$ **, a constant**

$\frac{\partial f}{\partial x} = y\, x^{y-1} f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\, (x^y\ ln\ x)$

$= y\, x^{y-1}\, ln\, x + x^y\, \frac{1}{x}$

$f_{yx} = \frac{\partial}{\partial y}\left(yx^{y-1}\right) = x^{y-1} + y\, x^{y-1}\, ln\, x$

Example:
$f(x, y) = sin(x - y) f_y\ = -cos(x - y)$
$f_x = cos(x - y) f_{yy} = -sin(x - y)$
$f_{xx} = -sin(x - y)$
Note that $f_{xx} - f_{yy} = 0.$ $\frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}\left(-cos(x - y)\right) = sin(x - y),$
$\frac{\partial}{\partial y} f_x = +sin(x - y).$

<u>Chain Rule for Partial Derivatives</u>: Recall that if $y = f(u)$ **and** $u = y(x)$. **Then**

$$\frac{dy}{dx} = \frac{dy}{du}\ \ \frac{du}{dx} = f'(u) g'(x) = f'(g(x)) y'(x).$$

Consider: $z = f(x, y)$ **and suppose** $x = g(r, s)$ $\ y = h(r, s)$. **i.e.,** x **and** y **are functions of the variables** r, s. **Then we have** $z = F(r, s) = f(g(r, s)),\ h(r, s))$. **Chain rule says**

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial r} = \frac{\partial F}{\partial r}$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial s} = \frac{\partial F}{\partial s}.$$

Example. $f(x, y) = e^{xy}$ $\quad x = r\, cos\theta$ $\quad y = r\, sin\theta$. **Find** f_r, f_θ **in terms of** r **and** θ.

$\frac{\partial f}{\partial r} = y\, e^{xy}\, cos\theta + (x\, e^{xy}) sin\theta = (re^{r^2 sin\theta cos\theta} sin\theta) cos\theta + rcos\theta\, e^{r^2 sin\theta cos\theta} sin\theta$

$\frac{\partial f}{\partial \theta} = r^2\, cos2\theta (e^{r^2} sin\theta cos\theta).$

<u>Leibnitz's Rule</u>: Often it is necessary to deal with a function $\phi(x)$ **defined by an integral of the form**

$$\phi(x) = \int_{A(x)}^{B(x)} f(x, t) dt$$

where f **is such that we cannot evaluate the integral. In particular, an expression for** $\phi'(x)$ **is often required. If** A **and** B **are finite constants, differentiation with respect to** x **under integral sign can be justified** $\forall \times\ \in (a, b)$ **when** f **and** $\frac{\partial f}{\partial x}$ **are continuous for a** $a \le x \le b$ **and** $A \le t \le B$. **More generally, when the limits are not constant we can think of** ϕ **as a function of** x **directly and also indirectly, through the intermediate variables** $A(x)$ **and** $B(x)$. **Hence write** $\phi = \phi(x, A, B)$ **and**

$$\frac{d\phi}{dx} = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial B}\frac{dB}{dx} + \frac{\partial\phi}{\partial A}\frac{dA}{dx}$$

$\frac{\partial\phi}{\partial x}$ is calculated by treating A and B as constants, i.e., merely by differentiating with respect to x under the integral sign. To evaluate the other partials of ϕ, let $F(x,t)$ be a function such that $f(x,t) = \frac{\partial F(x,t)}{\partial t}$. Then

$$\phi(x, A, B) = \int_A^B \frac{\partial F}{\partial t}\, dt = F(x,B) - F(x,A).$$

Therefore when x is held constant as A and B are imagined to vary $\Rightarrow$

$$\frac{\partial\phi}{\partial B} = \frac{\partial F(x,B)}{\partial B} = f(x,B) \qquad \frac{\partial\phi}{\partial A} = \frac{-\partial F(x,A)}{\partial A} = -f(x,A)$$

$$\Rightarrow \frac{d}{dx}\int_A^B f(x,t)dt = \int_A^B \frac{\partial f}{\partial x}(x,t)dt + f(x,B)\frac{dB}{dx} - f(x,A)\frac{dA}{dx}\,. \quad (*)$$

This is valid $\forall\, x \in (a,b)$ when f and also $A'(x)$ and $B'(x)$ are continuous.
(*) is known as Leibnitz's rule.

Example: If $y(x) = \int_a^x h(t)\, sin(x-t)dt \;\Rightarrow\; y'(x) = \int_a^x h(t)\, cos(x-t)dt$

and $y''(x) = -\int_a^x h(t)\, sin(x-t)dt + h(x)$.

Therefore it follows that $y(x)$ satisfies the differential equation

$$y''(x) + y(x) = h(x).$$

By setting $x = a$ in the expressions for y and $y'(x)$ we get $y(a) = 0$ $y'(a) = 0$.

In the case of a function defined by an improper integral

$$\phi(x) = \int_{A(x)}^{\infty} f(x,t)dt.$$

It may shown that $\frac{d}{dx}\int_{A(x)}^{\infty} f(x,t)dt = \int_A^{\infty} \frac{\partial f(x,t)dt}{\partial x} - f(x,A)\frac{dA}{dx}$.

Certain conditions must be met by f and $\frac{\partial f}{\partial x}$.

Example: $\phi(x) = \int_0^{\infty} e^{-t^2} cos(2tx)dt$

$$\frac{d\phi}{dx} = -2\int_0^{\infty} te^{-t^2} sin(2tx)dt$$

Integrating by parts $\Rightarrow$ $\frac{d\phi}{dx} = \left[e^{-t^2} sin(2tx)\right]_{t=0}^{t=\infty} - 2x\int_0^{\infty} e^{-t^2} cos(2tx)dt$

$$= -2x \int_0^\infty e^{-t^2} cos(2tx)dt$$

$\Rightarrow$ ϕ **satisfies the differential equation**

$$\frac{d\phi}{dx} + 2x\phi = 0$$

$\Rightarrow$ $\phi = ce^{-x^2}$. **When** $x = 0$ **the original expression for** ϕ **yields**

$$\phi(0) = \int_0^\infty e^{-t^2} dt = (\tfrac{1}{2})\sqrt{\pi} \Rightarrow \int_0^\infty e^{-t^2} cos(2tx)dt = (\tfrac{1}{2})\sqrt{\pi}\ e^{-x^2}$$

Introducing the change of variable $t = au$ **where** $a > 0$ **and writing** $x = \frac{b}{2a}$ **leads to**

$$\int_0^\infty e^{-a^2u^2} cos\, bu\, du = \frac{\sqrt{\pi}}{2a}\ e^{-b^2/4a^2}.$$