Ma 529
Lecture X

More consequences of the Cauchy Integral Theorem.

Theorem 2: If $f(z)$ is analytical throughout a simply connected domain D, then

$$F(z) = \int_{z_0}^{z} f(z)dz$$

is an analytic function whose derivative at each point of D is $f(z)$.

Theorem 3: If $f(z)$ is analytic in a simply connected domain D, then, provided the path of integration lies entirely in D, $\int_{z_0}^{z_1} f(z)dz = G(z_1) - G(z_o)$ where $G(z)$ is any antiderivative of $f(z)$.

Proof: By Theorem 2, since $f(z)$ is analytic in D, $F(z) = \int_{z_0}^{z} f(z)dz$ is an antiderivative of $f(z)$. If $G(z)$ is also an antiderivative then

$$F'(z) = G'(z) = f(z) \Rightarrow F(z) = G(z) + C \Rightarrow \int_{z_0}^{z} f(z)dz = G(z) + C.$$

Let $z = z_0 \Rightarrow G(z_0) = -C \Rightarrow$ result.

Example. What is $\int_0^{1+i\pi} (z^2 + cosh2z)dz =$?

$f(z) = z^2 + cosh2z$ is analytic everywhere.

Also, $G(z) = \frac{1}{3}z^3 + \frac{1}{2}sinh2z$ is an antiderivative. By Theorem 3:

$$\int_0^{1+i\pi} (z^2 + cosh2z)dz = \frac{1}{3}\, z^3 + \frac{1}{2}\, sinh2z\Big|_0^{1+i\pi}$$

$$= \frac{1}{3}(1+i\pi)^3 + \frac{1}{2}\, sinh2(1+i\pi)$$

$$= \frac{1}{3} - \pi^2 + \frac{sinh2}{2} + \frac{i\pi}{3}(3-\pi^2)$$

Theorem 4: If $u(x,y)$ is a solution of Laplace's equation in a domain D, then in D $\exists$ an analytic function having u as its real part, namely, $f(z) = u + iv$ where

$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} \{ -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy \}$ and the path of integration is from $z_0 = x_0 + iy_0$

to $z = x + iy$ lying entirely in D.

Theorem 5: (The Cauchy Integral Formula). Let $f(z)$ be analytic in a simply connected domain D. Let z_0 be any point in D and let C be any simply closed path in D enclosing z_0. Then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz.$$

Example: Evaluate $\oint_C \frac{e^z}{z^2+1} dz$ if C is

1) a unit circle with center at $z = i$.
2) a unit circle with center at $z = -i$.

We write $\oint_C \frac{e^z}{z^2+1} dz = \oint_C \frac{e^z}{z+i} \frac{dz}{z-i}$.

1) Call $z_0 = i$ and $f(z) = \frac{e^z}{z+i}$. By the Cauchy Integral formula we have:

$$\oint_C \frac{e^z}{z^2+1} dz = 2\pi i\, f(i) = 2\pi i\, \frac{e^i}{2i} = \pi(cos 1 + i sin 1).$$

2) Call $z_0 = -i$ and $f(z) = \frac{e^z}{z-i}$. Again

$$\oint_C \frac{e^z}{z^2+1} dz = 2\pi i\, f(-i) = 2\pi i\, \frac{e^{-i}}{-2i} = -\pi(cos 1 - i sin 1)$$

Theorem 6: (Cauchy Integral Formula for Higher Derivatives) Let $f(z)$ be analytic at z_0. Then $f(z)$ has derivatives of all orders at z_0, and the nth derivative at z_0 is:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

where C is any closed path about z_0 such that $f(z)$ is analytic on and inside C.

Remark: Functions of a real variable do not, in general, possess the derivative properties described by Theorem 6. For example $x^{\frac{7}{3}}$ possesses a first and second derivative at $x = 0$, but no derivative of higher order.

Theorem 7: (Morera's Theorem) If $f(z)$ is continuous in a domain D, and $\int_C f(z) dz = 0$ for every simple closed path C in D, then $f(z)$ is analytic in D.

Remark: Morera's Theorem is the converse of the Cauchy Integral Theorem.

<u>Complex sequences and series, Taylor and Laurent Expansions</u> -

Complex Sequences

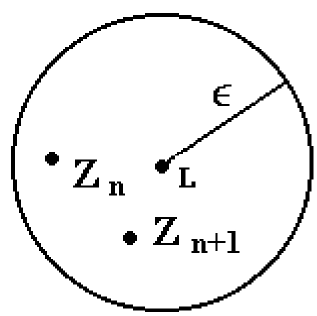
Definition: A complex sequence is a function from the positive integers to the complex numbers.

Notation: should be written $z(n)$ instead of z_n $n \geq 1$.

Example: $z_n = \frac{(-1)^n}{2n} + \frac{(3n-2)}{n}\, i$, $z_1 = \frac{-1}{2} + i$; $z_2 = \frac{1}{4} + 2i$; . . . etc.

Definition: A sequence $z_1, z_2 \ldots$ is said to *converge* to the number L if, given $\in\; > 0\; \exists$ an integer N such that $\mid z_n - L \mid\; < \;\in$ if $n \geq N$.

Remark: The definition means that given *any* $\in$ -neighborhood of $L \; \exists$ an N such that $z_n, z_{n+1}, \ldots$ are in this neighborhood. If $\{z_n\}$ converges to L we write $\lim\limits_{n \to \infty} z_n = L$ or $z_n \longrightarrow L$. If a sequence does not converge, we say it *diverges*.



Theorem: Let $z_n = x_n + iy_n$ be a sequence $(n \geq 1)$. Then $z_n \rightarrow A + iB \Leftrightarrow x_n \rightarrow A$ and $y_n \rightarrow B$.

Example: $z_n = \frac{(-1)^n}{2n} + \frac{(3n-2)}{n}$ i

$$x_n = \frac{(-1)^n}{2n} \text{ so } x_n \longrightarrow 0$$

$$y_n = 3 - \frac{2}{n} \text{ so } y_n \longrightarrow 3 \longrightarrow z_n \longrightarrow 3i$$

Cauchy's Converge Criterion

Definition: a complex sequence $z_1, z_2, \ldots$ is called a *Cauchy Sequence* if given $\in\ > 0, \exists$ an integer N such that: $|\, z_n - z_m \,| < \in$ if $n \geq N,\ \ m \geq N.$

Theorem: (Cauchy Convergence Criterion) A complex sequence $z_1, z_2, \ldots$ converges $\Leftrightarrow$ it is a Cauchy Sequence.

Complex Series. Consider a series of complex constants $\alpha_1 + \alpha_2 + \alpha_3 + \ldots$. Let

$S_m = \sum_{n=1}^{m} \alpha_n = \alpha_1 + \alpha_2 + \ldots + \alpha_m$ (m the partial sum).

Definition: The series $\sum_{n=1}^{\infty} \alpha_n$ is said to converge to a number S if the sequence of partial sums $S_1, S_2, \ldots,$ converges to S. In this case we write $\sum_{n=1}^{\infty} \alpha_n = S$. If the series does not converge, we say it diverges.

Theorem: Let $\alpha_n = b_n + ic_n$ (b_n, c_n real). $\sum_{n=1}^{\infty} \alpha_n$ converges $\Leftrightarrow$ each of the real series $\sum_{1}^{\infty} b_n$ and $\sum_{1}^{\infty} c_n$ converges.

Definition: The series $\sum_{1}^{\infty} \alpha_n$ is said to converge in the Cauchy sense if the sequence of partial sums $S_1, S_2, \ldots$ is a Cauchy Sequence.

Theorem: $\sum_{n=1}^{\infty} \alpha_n$ is convergent (i.e. converges to a number) $\Leftrightarrow$ it is Cauchy convergent.

Theorem: If $\sum_{n=1}^{\infty} \alpha_n$ converges, then $\lim_{n \to \infty} \alpha_n = 0$.

Example: $\sum_{n=1}^{\infty} \frac{1}{n^2} + in^2$ Since $\lim_{n \to \infty} \frac{1}{n^2} + in^2 \neq 0$ therefore the series diverges.

Remark: Just because $\alpha_n \to 0$, this does not mean $\sum \alpha_n$ converges.

Example: $\sum_{1}^{\infty} \frac{1}{n}$ $\alpha_n = \frac{1}{n}$ $\alpha_n \to 0$, but $\sum \frac{1}{n}$ diverges.

Definition: $\sum_{1}^{\infty} \alpha_n$ is said to be <u>absolutely convergent</u> if the series $\sum_{n=1}^{\infty} |\, \alpha_n \,|$ converges.

Remark: Absolute convergence implies ordinary convergence ;

Ordinary convergence does *not* imply absolute convergence.

Example: $\sum\limits_{1}^{\infty} \frac{(-1)^n}{n}$ **converges, but** $\sum \left| \frac{(-1)^n}{n} \right| = \sum \frac{1}{n}$ **diverges.**

Consider now functions $f_1(z)$, $f_2(z), \ldots$ **all defined in a region** R **and the series** $\sum\limits_{n=1}^{\infty} f_n(z)$. **For each** z **this is a series.**

Theorem: (Ratio Test) For the series $\sum\limits_{n=1}^{\infty} f_n(z)$ **let**

$$\lim_{n \to \infty} \frac{f_{n+1}(z)}{f_n(z)} = | \, R(z) \, | \, .$$

Then the given series converges absolutely for those values of z **for which** $0 \leq | \, R(z) \, | < 1$; **and diverges for those values of** z **for which** $| \, R(z) \, | > 1$.

Remark: The values of z **for which** $| \, R(z) \, | = 1$ **form the boundary of the region of convergence (set of all values of** z **for which the series converges). At these points the ratio test provides no information about the convergence of divergence of the series.**

Example: Find the region of convergence of the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{z+1}{z-1} \right)^{n-1} = 1 + \frac{1}{2^2} \left(\frac{z+1}{z-1} \right) + \frac{1}{3^2} \left(\frac{z+1}{z-1} \right)^2 + \ldots .$$

Using the ratio test we have

$$\left| \frac{f_{n+1}(z)}{f_n(z)} \right| = \left| \frac{\frac{1}{(n+1)^2} \left(\frac{z+1}{z-1} \right)^n}{\frac{1}{n^2} \left(\frac{z+1}{z-1} \right)^{n-1}} \right| = \frac{n}{(n+1)} \left| \frac{z+1}{z-1} \right| .$$

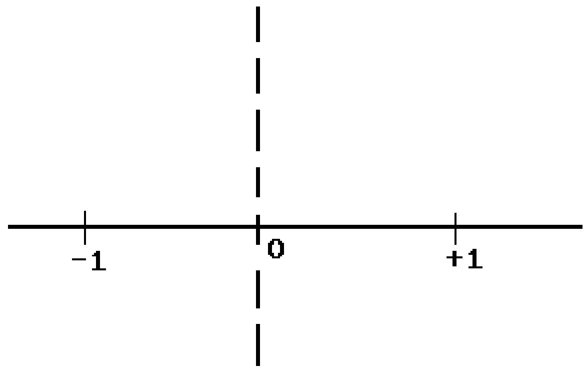
Now

$$\lim_{n \to \infty} \left| \frac{f_{n+1}(z)}{f_n(z)} \right| = \left| \frac{z+1}{z-1} \right| = | \, R(z) \, | \, .$$

Therefore the series certainly converges if

$$\left| \frac{z+1}{z-1} \right| < 1, \text{ i.e., } | \, z+1 \, | < | \, z-1 \, | \, .$$

Therefore z must lie to the left of the $\perp$ bisector of the segment joining -1 and $+1$; i.e., z must lie in the left half of the complex plane.



Test gives no information on the boundary, i.e., for values of z on the imaginary axis. For $z = ai$ (a any constant), we have for absolute value of the series that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \left| \frac{z+1}{z-1} \right|^{n-1} = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ which converges.}$$

Since absolute convergence $\longrightarrow$ convergence $\longrightarrow$ series converges also on the imaginary axis. Therefore series converges $\forall z = x + yi$ such that $x \leq 0$.

<u>Uniform convergence</u>: Let $f_1(z)$, $f_2(z)$... all be defined in a region R and $S_n(z) = \sum_{i=1}^{n} f_i(z)$.

Definition: The series $\sum_{i=1}^{\infty} f_i(z)$ is said to <u>converge uniformly</u> to the function $f(z)$ if, given $\in > 0$ $\exists$ a positive integer $N = N(\in)$ such that $|\, f(z) - S_n(z) \,| < \in$ for any $n > N$ and any $z \in R$.

Note: If convergence is *not* uniform, then $N = N(\in, z)$.

Theorem: If $f_1(z)$, $f_2(z)$... are all continuous in R and if $\sum f_i(z)$ converges uniformly in R to $f(z)$, then the sum $f(z)$ is also a continuous function in R.