Ma 529
Lecture XII

Laurent Series

In many applications it is necessary to expand functions around points at which, or in the neighborhood of which, the functions are not analytic. The method of Taylor Series is not applicable in such cases, and a new type of series known as Laurent Series is required. This furnishes us with a representation which is valid in the annular ring bounded by two concentric circles, provided that the function being expanded is analytic everywhere between the two circles. The price we pay for this is that negative powers of $z - z_0$ as well as positive powers appear in the expansion.

Laurent's Theorem: If $f(x)$ is analytic (and singled-valued) in the annular ring $0 < r < |z - z_0| < R$, it can be expanded there into a series of the form

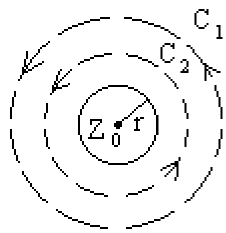
$$(1)\ \ f(x) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where

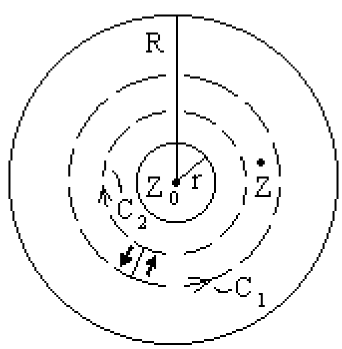
$$(2)\ \ a_n = \frac{1}{2\pi i} \oint_C \frac{f(t)dt}{(t - z_0)^{n+1}} \quad n = 0, \pm 1, \pm 2, ...$$

and C is $|z - z_0| = \rho, \quad r < \rho < R$.

Proof: Fix z. Pick C_1 and C_2 of radii r_1 and r_2 respectively such that $r < r_2 < |z - z_0| < r_1 < R$.



By the Cauchy Integral Formula for the path shown below



we have

$$f(z) = \frac{1}{2\pi i} \int_{C_1 - C_2} \frac{f(t)}{t-z} dt$$

$$f(z) = \frac{1}{2\pi i} \left\{ \int_{C_1} \frac{f(t)}{t-z} dt - \int_{C_2} \frac{f(t)}{t-z} dt \right\}.$$

On C_1: $\frac{1}{t-z} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(t-z_0)^{n+1}}$

This converges because $|z - z_0| < |t - z_0|$. (This is similar to what we did before).

On C_2: $-\frac{1}{t-z} = \frac{1}{z-z_0-(t-z_0)} = \sum_{n=0}^{\infty} \frac{(t-z_0)^n}{(z-z_0)^{n+1}} = \sum_{n=1}^{\infty} \frac{(t-z_0)^{n-1}}{(z-z_0)^n}$.

This converges because $|t - z_0| < |z - z_0|$.

$$-\frac{1}{t-z} = \sum_{j=-1}^{-\infty} \frac{(t-z_0)^{-j-1}}{(z-z_0)^{-j}} = \sum_{j=-1}^{-\infty} \frac{(z-z_0)^j}{(t-z_0)^{j+1}} = \sum_{n=-1}^{-\infty} \frac{(z-z_0)^n}{(t-z_0)^{n+1}} .$$

Therefore

$$f(z) = \frac{1}{2\pi i} \left\{ \sum_{n=0}^{\infty} \oint_{C_1} \frac{f(t)dt}{(t-z_0)^{n+1}}(z - z_0)^n + \sum_{n=-1}^{-\infty} \oint_{C_2} \frac{f(t)dt}{(t-z_0)^{n+1}}(z - z_0)^n \right\}.$$

The above is true if we deform C_1 and C_2 into C: $|z - z_0| = \rho$. Thus

$$f(z) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} (z - z_0)^n \oint_{C} \frac{f(t)dt}{(t-z)^{n+1}}$$

as required.

Remark: Laurent's Theorem says that a function $f(z)$ analytic in $r < |z - z_0| < R$ may be represented as $f(z) = f_1(z) + f_2(z)$ where $f_1(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ is analytic for $|z - z_0| < R$ and $f_2(z) = \sum_{n=-1}^{-\infty} a_n(z - z_0)^n = \sum_{n=1}^{\infty} \frac{a_n}{(z-z_0)^n}$ is regular for $|z - z_0| > r$.

Thus any problems experienced by $f(z)$ at z_0 are contained in $f_2(z)$. If $f(z)$ is analytic at z_0, then $f(z) = f_1(z), f_2(z) = 0 \rightarrow$ Laurent expansion about z_o reduces to the Taylor expansion about z_0.

Example: Find the Laurent expansion of the function $f(z) = \frac{7z-2}{(z+1)z(z-2)}$ in the annulus $1 < |z + 1| < 3$.

First we apply the method of partial fractions to $f(z)$ and get

$$f(z) = \frac{-3}{z+1} + \frac{1}{z} + \frac{2}{z-2}.$$

Now the center of the annulus is $z = -1$ and therefore the series we are seeking must be one involving powers of $z+1$. Therefore we modify the second and third terms in the partial fraction representation of $f(z)$ so that z will appear in the combination $z+1$. Thus

$$f(z) = \tfrac{-3}{z+1} + \tfrac{1}{(z+1)-1} + \tfrac{2}{(z+1)-3}$$

$$f(z) = -3(z+1)^{-1} + \tfrac{1}{(z+1)-1} + \tfrac{2}{(z+1)-3}\,.$$

Now $\frac{1}{1-t} = \sum_{o}^{\infty} t^n$ **for** $|\,t\,| < 1$. **If we consider** $\frac{1}{(z+1)-1}$ **as** $\frac{-1}{1-(z+1)}$ **We get**

$$\tfrac{-1}{1-(z+1)} = -\sum_{0}^{\infty}(z+1)^n$$

provided $|\,z+1\,| < 1$, **but this is *not* in our annulus! Hence** $1 < |\,z+1\,|$ **so**

$$\tfrac{1}{(z+1)-1} = \tfrac{1}{(z+1)\left[1-\frac{1}{z+1}\right]} = \tfrac{1}{z+1}\sum_{0}^{\infty}\left(\tfrac{1}{z+1}\right)^n = \sum_{0}^{\infty}\tfrac{1}{(z+1)^{n+1}}.$$

For $\frac{2}{(z+1)-3}$ **we write**

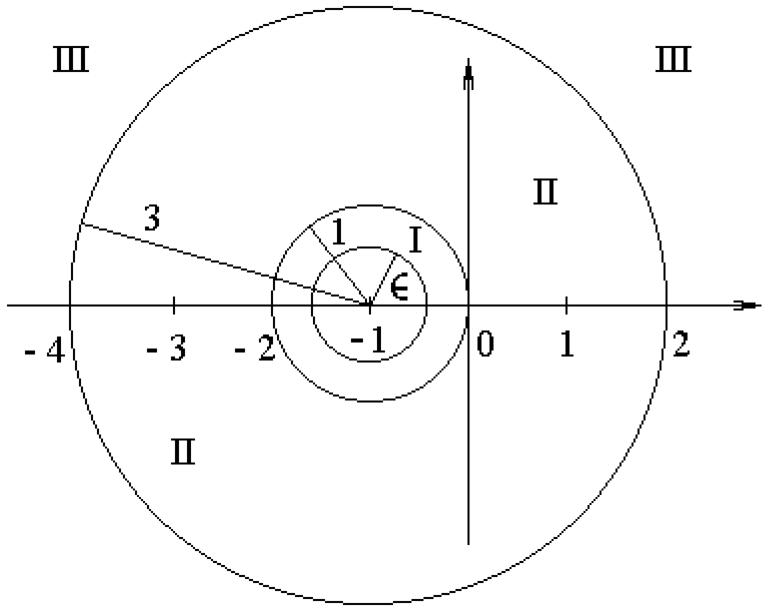
$$\tfrac{-2}{3-(z+1)} = \tfrac{-2}{3\left[1-\frac{(z+1)}{3}\right]} = \tfrac{-2}{3}\sum_{0}^{\infty}\left(\tfrac{z+1}{3}\right)^n.$$

Therefore

$$f(z) = -3(z+1)^{-1} + \sum_{0}^{\infty}\tfrac{1}{(z+1)^{n+1}} - 2\sum_{0}^{\infty}\tfrac{(z+1)^n}{3^{n+1}}$$
$$= \cdots + \tfrac{1}{(z+1)^3} + \tfrac{1}{(z+1)^2} - \tfrac{2}{z+1} - \tfrac{2}{3} - \tfrac{2}{9}(z+1) - \tfrac{2}{27}(z+1)^2 + \cdots$$

provided $|\,z+1\,| < 3$.

It is important to note that $f(z)$ has two other Laurent expansions around the point $z = -1$. One is valid in the annulus region between a circle of arbitrary small radius around $z = -1$. The other is valid in the region exterior to a circle of radius 3 around $z = -1$. Regions of validity of the three Laurent expansions of $\frac{7z-2}{(z+1)z(z-2)}$ around $z = -1$.



Each of these expansions can be found, as above, by suitably rearranging the terms in the partial fraction representation of $f(z)$ and then expanding these terms by means of the geometric series expansion. Thus in the innermost region I we have

$$f(z) = -3(z+1)^{-1} + [-1 + (z+1)]^{-1} + 2[-3 + (z+1)]^{-1}$$

$$= -3(z+1)^{-1} - \sum_0^\infty (z+1)^n - 2\sum_0^\infty \frac{(z+1)^n}{3^{n+1}}$$

$$= -3(z+1)^{-1} - \frac{5}{3} - \frac{11}{9}(z+1) - \frac{29}{27}(z+1)^2 - \frac{83}{81}(z+1)^3 - \cdots$$

provided $0 < |z+1| < 1$.

Similarly in the outermost region III we have

$$f(z) = -3(z+1)^{-1} + [(z+1) - 1]^{-1} + 2[(z+1) - 3]^{-1}$$

$$= -3(z+1)^{-1} + \sum_0^\infty \frac{1}{(z+1)^{n+1}} + 2\sum_0^\infty \frac{3^n}{(z+1)^{n+1}}$$

$$= \cdots + 19(z+1)^{-3} + 7(z+1)^{-2}$$

provided $|z+1| > 3$.

Remark: We have obtained these Laurent expansions without using the general theory. Thus we can evaluate the integrals in the coefficient formulas by comparing them with the numerical values of the coefficients we have found by other means. For instance, in the

first expansion of $f(z)$ $[1 < |z+1| < 3]$ the coefficient of $(z+1)^{-1}$ is -2. Thus from

$$\text{(2)}\ \ a_n = \frac{1}{2\pi i} \oint_C \frac{f(t)dt}{(t-z_0)^{n+1}} \qquad n = 0, \pm 1, \pm 2, ...$$

in the theorem we have

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(t)dt.$$

For our example $a_{-1} = \frac{1}{2\pi i} \oint_C \frac{7t-2}{(t+1)t(t-2)}\,dt$

where C is any closed curve lying in the interior of the circle $|z+1| = 3$ and enclosing the circle $|z+1| = 1$. Thus, without doing any integration we have

$$\frac{1}{2\pi i} \oint_C \frac{7z-2}{(z+1)z(z-2)} dz = -2$$

or

$$\oint_C \frac{7z-2}{(z+1)z(z-2)} dz = -4\pi i.$$

We could not use the Cauchy Integral Formula to get this.

Singularities, Residues, and Applications to Real Integrals and Series

Isolated Singularities

Suppose that $f(z)$ is regular and single-valued for $0 < |z-z_0| < R$. Then we have the expansion

$$\text{(1)} \qquad f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

which is convergent for $0 < |z-z_0| < R$. If all a_{-n} $(n = 1, 2, ...)$ are zero, then (1) reduces to a Taylor series convergent for $0 \le |z-z_0| < R$, i.e., $f(z)$ is analytic at $z = z_0$.

If not all of the a_{-n} are zero, then $z = z_0$ is a singular point of $f(z)$. We say in this case that $z = z_0$ is an *isolated singularity* of $f(z)$.

Example: $f(z) = \frac{1}{sin\left(\frac{1}{z}\right)}$ **has a singularity at $z = 0$ which is not isolated, since**

$$sin\left(\frac{1}{z}\right) = 0 \qquad \text{at } \frac{1}{z} = n\pi \text{ or } z = \frac{1}{n\pi} \quad n = 0, \pm 1, ...$$

Example: $f(z) = \frac{1}{sin z}$ **has an isolated singularity at** $z = 0$.

Classification of Isolated Singularities

1) Poles: If $f(z) = \frac{a_{-m}}{(z - z_0)^m} + \cdots + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + \cdots$

where $a_{-m} \neq 0$ **and** $a_{-n} = 0$ **for** $n \geq m + 1$, $f(z)$ **is said to have a pole of order** m **at** $z = z_0$. **The rational function**

$$\frac{a_{-m}}{(z - z_0)^m} + \cdots + \frac{a_{-1}}{(z - z_0)}$$

is called the *principal part* or *meromorphic part* of $f(z)$ **at** $z = z_0$.

2) If the Laurent series (1) contains infinitely many non-vanishing a_{-n}, $f(z)$ **is said to have an *essential singularity* at** $z = z_0$.

Remark: If $z = z_0$ **is a pole of** $f(z)$**, then as** $z \to z_0$ $f(z) \to \infty$ **;** $f(z_0) = \infty$. **If** $f(z)$ **has a pole of order** m **at** $z = z_0$**, then** $\frac{1}{f(z)}$ **is regular at** $z = z_0$ **and has there a zero of order** m.

$$\frac{1}{f(z)} = (z - z_0)^m \; g(z)$$

where $g(z)$ **is regular at** $z = z_0$. $g(z_0) \neq 0$.

Example: $\frac{1}{z(z-1)^2} = \frac{1}{[1+(z-1)](z-1)^2}$ **about** $z = 1$

$$= \frac{1}{(z-1)^2}\left[\frac{1}{1-[-(z-1)]}\right] = \frac{1}{(z-1)^2}\sum_{0}^{\infty}(-1)^n(z-1)^n$$

$$= \sum_{0}^{\infty}(-1)^n(z-1)^{n-2} \quad 0 < |z-1| < 1$$

$$= \frac{1}{(z-1)^2} - \frac{1}{(z-1)} + 1 - (z-1) + \cdots.$$

Thus $\frac{1}{z(z-1)^2}$ **has a pole of order 2 at** $z = 1$ **and its principal part there is** $\frac{1}{(z-1)^2} - \frac{1}{(z-1)}$.

Remark: We can also write:

$$\frac{1}{z(z-1)^2} = \frac{1}{(z-1)^2}\left[\frac{1}{(z-1)+1}\right] = \frac{1}{(z-1)^3}\left[\frac{1}{1-\left(\frac{-1}{z-1}\right)}\right]$$

$$= \frac{1}{(z-1)^3}\sum_{0}^{\infty}\frac{(-1)^n}{(z-1)^n} = \sum_{0}^{\infty}\frac{(-1)^n}{(z-1)^{n+3}} \quad \text{for } \left|\frac{1}{z-1}\right| < 1 \text{ or } 1 < |z-1|$$

$= \cdots + \frac{1}{(z-1)^5} - \frac{1}{(z-1)^4} + \frac{1}{(z-1)^3}$ (*)

This expansion contains infinitely many negative powers of $z-1$. This does not contradict our observation that $\frac{1}{z(z-1)^2}$ has a pole of order 2 at $z=1$. Reason: The series (*) is valid only *outside* the circle $|z-1| = 1$, whereas the presence of poles and essential singularities is determined by the particular Laurent expansion which is valid in the *innermost* annulus, or deleted neighborhood, of the point in question.

Example: $e^{\frac{1}{z}}$ near $z=0$. Since $e^{\frac{1}{z}} = 1 + \frac{1}{2} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$ it has an essential singularity at the origin.

Remark: If $f(z)$ has a Taylor series near $z=z_0$ (no negative powers of $(z-z_0)$ in the Laurent expansion) , then $f(z)$ is said to have a *removable singularity* at $z=z_0$.

Example: $\frac{sinz}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$ Thus the singularity is removable at $z=0$.

The Residue Theorem

Recall $a_{-n} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{-n+1}} dz \quad n = 1, 2, \ldots$ from Laurent's Theorem. Thus

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)dz.$$

Theorem: Suppose $f(z)$ is analytic on the simple, closed contour C and in its interior, except at $z = z_1, z_2, \ldots, z_n$ which are isolated singularities of $f(z)$. Then

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{n} \underset{z_j}{Res}\, f(z)$$

where $\underset{z_j}{Res}\, f(z)$ is the residue of $f(z)$ at $z = z_j$ $(j = 1, \ldots, n)$.

Definition: Suppose $f(z)$ has an isolated singularity at $z=z_0$ and is single-valued in a neighborhood of $z=z_0$; C is a simple, closed contour about $z=z_0$ and $f(z)$ is analytic on C and throughout its interior except at $z=z_0$. Then a_{-1} is called the *residue* of $f(z)$ at $z=z_0$.

Simple Pole: $f(z) = \frac{\underset{z_0}{Res}\, f(z)}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$ Hence

$$(z-z_o)f(z) = \underset{z_0}{Res}\, f(z) + a_0(z-z_0) + a_1(z-z_0)^2 + \cdots$$

Hence $$\underset{z_0}{Res}\, f(z) = \lim_{z \to z_0} (z-z_0)f(z)$$

Double Pole: $f(z) = \frac{a_{-2}}{(z-z_0)^2} + \frac{\underset{z_0}{Res}\, f(z)}{(z-z_0)} + a_0 + \cdots$

$$(z-z_0)^2 f(z) = a_{-2} + \underset{z_0}{Res}\, f(z)(z-z_0) + a_0(z-z_0)^2 + \cdots$$

$$\frac{d}{dz}\left[(z-z_0)^2 f(z)\right] = \underset{z_0}{Res}\, f(z) + 2a_o(z-z_0) + \cdots$$

Hence $\underset{z_0}{Res}\, f(z) = \lim_{z \to z_0} \frac{d}{dz}\left[(z-z_0)^2 f(z)\right].$

nth Order Pole: $\underset{z_0}{Res}\, f(z) = \lim_{z \to z_0} \frac{\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right]}{(n-1)!}$.

Example: $f(z) = \frac{z^2-2z}{z^2-1} - \frac{z^2-2z}{(z+1)(z-1)}$. **Simple poles at** $z = \pm 1$.

$\underset{+1}{Res}\, f(z) = \lim_{z \to 1} \frac{z^2-2z}{z+1} = -\frac{1}{2}$; **and** $\underset{-1}{Res}\, f(z) = \lim_{z \to -1} \frac{z^2-2z}{z-1} = -\frac{3}{2}$.

Example: $\frac{z^3+5}{z(z-1)^3}$. **Triple pole at** $z = 1$.

$$\underset{1}{Res}\, \frac{z^3+5}{z(z-1)^3} = \lim_{z \to 1} \frac{1}{2!}\left[\frac{d^2}{dz^2}\left(\frac{z^3+5}{z}\right)\right] = \lim_{z \to 1} \frac{1}{2!}\left[\frac{d^2}{dz^2}\,(z^2 + \tfrac{5}{z})\right] = \lim_{z \to 1} \frac{1}{2!}\frac{d}{dz}\left[2z - \tfrac{5}{z^2}\right]$$

$$= \lim_{z \to 1} \frac{1}{2!}\left[2 + \tfrac{10}{z^3}\right] = \frac{12}{2} = 6.$$

Example: Find $\oint_C f(z)dz$ **where** $f(z) = \frac{z^3+5}{z(z-1)^3}$ **and** C **is the circle** $|z| = 2$.

The integrand has poles at $z = 0$ **and** $z = 1$. $\underset{1}{Res} = 6$; $\underset{0}{Res} = \left.\frac{z^3+5}{(z-1)^3}\right|_{z=0} = -5$. **Therefore**

$$\oint_C f(z)dz = 2\pi i(6-5) = 2\pi i.$$

Example: What is the integral of $f(z) = \frac{-3z+4}{z(z-1)(z-2)}$ **around the circle** $|z| = \frac{3}{2}$ **?**

There are poles at $z = 0, 1, 2$. **However, 2 is outside the circle and we need not consider it. At** $z = 0$ **and** $z = 1$ **we have simple poles, so that**

$$\underset{0}{Res}\, f(z) = \frac{4}{2} = 2 \text{ ; and } \underset{1}{Res}\, f(z) = \frac{1}{-1} = -1$$

therefore $\oint_C f(z)dz = 2\pi i(2-1) = 2\pi i.$

Notice that we are able to evaluate the integrals in the last two examples without doing any actual integration.