Ma 529
Lecture XIII

Application of the Residue Theorem to the Evaluation of Real Integrals

A. $$I = \int_0^{2\pi} R(cos\theta, sin\theta)d\theta.$$

Here $R(s,t)$ is a rational function of s and t. We assume that $R(cos\theta, sin\theta)$ is finite on $0 \le \theta \le 2\pi$.

Let $z = e^{i\theta}$. Then the interval $0 \le \theta \le 2\pi$ yields the curve C: $|z| = 1$.

Now $cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + \frac{1}{z})$ and $sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$; also $dz = ie^{i\theta}d\theta = izd\theta$ or $d\theta = (\frac{1}{i})(\frac{dz}{z})$. Thus

$$I = \oint_C R\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right] \quad \text{where } C: |z| = 1.$$

I may be evaluated by residues.

Example: Evaluate $\int_0^{2\pi} \frac{cos2\theta d\theta}{1-2p\,cos\theta+p^2} \quad -1 < p < 1.$

Note that by adding and subtracting $2p$, the denominator of the integrand can be written in either of two equivalent forms:

$$\begin{aligned} 1 - 2p\,cos\theta + p^2 &= 1 - 2p + p^2 + 2p - 2p\,cos\theta = (1-p)^2 + 2p(1 - cos\theta) \\ &= 1 + 2p + p^2 - 2p - 2p\,cos\theta = (1+p)^2 - 2p(1 + cos\theta). \end{aligned}$$

From the first of these it is clear that if $0 \le p < 1$, the denominator is different from zero for all values of θ; and from the second it is clear that if $-1 < p \le 0$, the denominator is also different from zero for all values of θ. Therefore the integrand is finite on $0 \le \theta \le 2\pi$ for $-1 < p < 1$. Now

$$cos2\theta = \frac{e^{2i\theta}+e^{-2i\theta}}{2} = \frac{z^2+z^{-2}}{2}.$$

Therefore the integral becomes:

$$\begin{aligned} &\oint_C \frac{z^2-z^{-2}}{2} \; \frac{1}{1-\frac{2p(z+z^{-1})}{2}+p^2} \; \frac{dz}{iz} \\ &= \oint_C \frac{z^4+1}{2z^2} \; \frac{z}{z-pz^2-p+p^2z} \; \frac{dz}{iz} \end{aligned}$$

$$= \oint_C \frac{(1+z^4)dz}{2iz^2(1-pz)(z-p)} \, .$$

There are three poles in the integrand at 0, $\frac{1}{p}$, and p. Now $-1 < p < 1$

so that $\frac{1}{p}$ is outside of $|z| = 1$. Therefore we may disregard this pole. For the other two poles we have:

$$\underset{p}{Res} = \lim_{z \to p} (z-p) \frac{1+z^4}{2iz^2(1-pz)(z-p)} = \frac{1+p^4}{2ip^2(1-p^2)} \, .$$

The pole at 0 is second order and therefore

$$\underset{p}{Res} = \lim_{z \to 0} \frac{d}{dz} \left[z^2 \frac{1+z^4}{2iz^2(z-pz^2-p+p^2z)} \right]$$

$$= \lim_{z \to 0} \frac{(z-pz^2-p+p^2z)(4z^3)-(1+z^4)(1-2pz+p^2)}{2i(z-pz^2-p+p^2z)^2}$$

$$= -\left(\frac{1+p^2}{2ip^2}\right)$$

$$I = 2\pi i \left[\frac{1+p^4}{2ip^2(1-p^2)} - \frac{1+p^2}{2ip^2}\right] = \frac{2\pi p^2}{1-p^2} \, .$$

B. <u>Improper Integrals of Rational Functions</u>

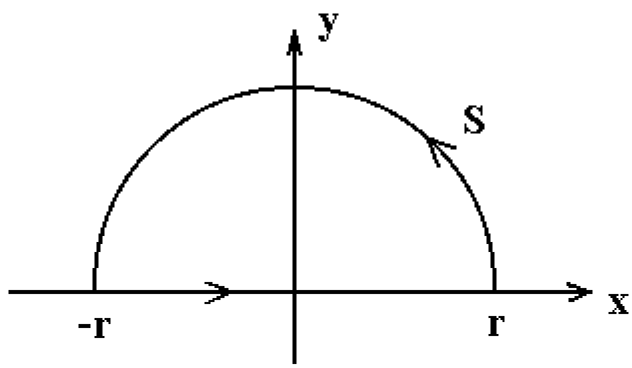
Consider $\int_{-\infty}^{\infty} f(x)dx$ where $f(x) = \frac{P(x)}{Q(x)}$ is a rational function and the degree of $[Q(x)] \geq [P(x)] + 2$. Suppose $Q(x)$ has no real zeros. Such an integral is called an improper integral since the interval of integration is not finite. We define

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r \to \infty} \int_{-r}^{r} f(x)dx.$$

To evaluate this integral consider the contour integral

$$\oint_C f(z)dz$$

where C is below.



Since $f(x)$ is a rational function, this implies that $f(z) = \frac{P(z)}{Q(z)}$ has finitely many poles in the upper half plane. These are the zeros of $Q(z)$, say $z_1, \ldots, z_n$. We choose r large enough so that C encloses all of these zeros. Then

$$\oint_C f(z)dz = \int_S f(z)dz + \int_{-r}^{r} f(x)dx = 2\pi i \sum_{j=1}^{n} \underset{z_j}{Res}\, f(z)\,.$$

Therefore

$$\int_{-r}^{r} f(x)dx = 2\pi i \sum_{j=1}^{n} \underset{z_j}{Res}\, f(z) - \int_S f(z)dz.$$

It turns out that $\int_S f(z)dz \longrightarrow 0$ as $r \longrightarrow \infty$. On S, if we let $z = re^{i\theta}$, then S is represented by $r = constant, 0 \leq \theta \leq \pi$. Since $deg[Q(z)] \geq deg[P(z)] + 2$. Hence

$$\mid f(z) \mid \; < \frac{k}{|z|^2} \text{ for } \mid z \mid \; = r > r_0,$$

where r_0 and k are sufficiently large. Hence

$$\left| \int_S f(z)dz \right| < \frac{k}{r^2}\pi r = \frac{k\pi}{r} \quad \text{for } r > r_0$$

and this $\longrightarrow$ 0 as $r \longrightarrow \infty$, so that

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{j=1}^{n} \underset{z_j}{Res}\, f(z)\,.$$

Example: $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx$ where $a, b > 0$.

The poles of $\frac{z^2}{(z^2+a^2)(z^2+b^2)}$ are at $z = \pm ai, \pm bi$. Of these only $z = ai$ and $z = bi$ lie in the upper half plane. At $z = ai$ the residue is:

$$\lim_{z \to ai} (z - ai) \frac{z^2}{(z-ai)(z+ai)(z^2+b^2)} = \frac{-a^2}{2ai(-a^2+b^2)} = \frac{a}{2i(a^2-b^2)}\ .$$

Similarly, at $z = bi$ we have $\frac{b}{2i(b^2-a^2)}$.

Hence the value of the integral is: $2\pi i \left[\frac{a}{2i(a^2-b^2)} + \frac{b}{2i(b^2-a^2)}\right] = \frac{\pi}{a+b}$.

C. $\int_{-\infty}^{\infty} f(x) cos\ ax\ dx$ and $\int_{-\infty}^{\infty} f(x) sin\ ax\ dx$. Here f is as in B.

Note that

$$\int_{-\infty}^{\infty} f(x) cos\ ax\ dx + i \int_{-\infty}^{\infty} f(x) sin\ ax\ dx = \int_{-\infty}^{\infty} f(x) e^{iax} dx.$$

The $\int_{-\infty}^{\infty} f(x) e^{iax} dx$ may be treated as above. Thus

$\int_{-\infty}^{\infty} f(x) e^{iax} dx = 2\pi i \sum$ Residues of $e^{iax} f(x)$ at its poles in the upper half plane.

Therefore

$\int_{-\infty}^{\infty} f(x) cos\ ax\ dx$
$= -2\pi \sum$ imaginary parts of residues of $e^{iax} f(z)$ at its poles in upper half plane.

Similarly

$\int_{-\infty}^{\infty} f(x) sin\ ax\ dx$
$= 2\pi \sum$ real parts of the residues of $e^{iax} f(z)$ at its poles in upper half plane.

Example. $\int_{-\infty}^{\infty} \frac{cosaxdx}{k^2+x^2} = \frac{\pi}{k} e^{-ak}$; $\int_{-\infty}^{\infty} \frac{sinaxdx}{k^2+x^2} = 0$ $\quad a > 0,\ k > 0.$

Now $\frac{e^{iax}}{k^2+z^2}$ has only one pole in the upper half plane, namely a single pole at $z = ik$. Therefore

$$\operatorname*{Res}_{z = ik} \frac{e^{iaz}}{k^2+z^2} = \lim_{z \to ik} (z - ik) \frac{e^{iaz}}{(z-ik)(z+ik)} = \frac{e^{-ka}}{2ik}.$$

Therefore

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{k^2+x^2} dx = 2\pi i \, \frac{e^{-ka}}{2ik} = \frac{\pi}{k} \, e^{-ka}$$

and this yields the result.

Laplace Transforms

The Laplace Transform of a (real) function $f(t)$ is given by

$$\mathcal{L}[f(t)] = \phi(s) = \int_0^{\infty} e^{-st} f(t) dt$$

for all s such that this integral converges.

Example. $f(t) = e^{at}$

$$\mathcal{L}[f(t)] = \mathcal{L}[e^{at}] = \int_0^{\infty} e^{-st} e^{at} \ dt = \int_0^{\infty} e^{-(s-a)^t} dt$$

$$= \lim_{R \to \infty} \int_0^{R} e^{-(s-a)t} \, dt = \lim_{R \to \infty} \frac{e^{-(s-a)t}}{a-s} \Big|_0^R$$

$$= \lim_{R \to \infty} \frac{e^{-(s-a)R} - 1}{a-s} = \frac{1}{s-a} \ \text{ if } \ s > a.$$

Therefore $\mathcal{L}[e^{at}] = \frac{1}{s-a}$.

We may define an inverse operator $\mathcal{L}^{-1}$ **where** $\mathcal{L}^{-1}[\phi(s)] = f(t) \Leftrightarrow \mathcal{L}[f(t)] = \phi(s)$. **Thus**

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}.$$

Laplace transforms are very useful in solving ordinary and partial differential equations. To illustrate the power of the method, consider the problem:

$$\frac{dy}{dt} + y = e^{2t}; \ y(0) = -1.$$

We take Laplace transforms of both sides of the equation to get

$$\mathcal{L}\left[\frac{dy}{dt}\right] + \mathcal{L}[y] = \mathcal{L}[e^{2t}] = \frac{1}{s-2}$$

Now $\mathcal{L}\{\frac{dy}{dt}\} = \int_0^{\infty} \frac{dy}{dt} e^{-st} dt = y e^{-st} \Big|_0^{\infty} + s \int_0^{\infty} y e^{-st} dt.$
Let $dv = \frac{dy}{dt} dt$ **and** $u = e^{-st}$ **so that** $du = -s e^{-st} dt$ **and** $v = y$. **Then**

$\mathcal{L}[y'] = -y(0) + s\mathcal{L}[y]$. **The D.E. then leads to**

$$(s+1)\mathcal{L}[y] + 1 = \frac{1}{s-2} \quad \frac{1}{s-2} - 1 = \frac{-s+3}{s-2}$$

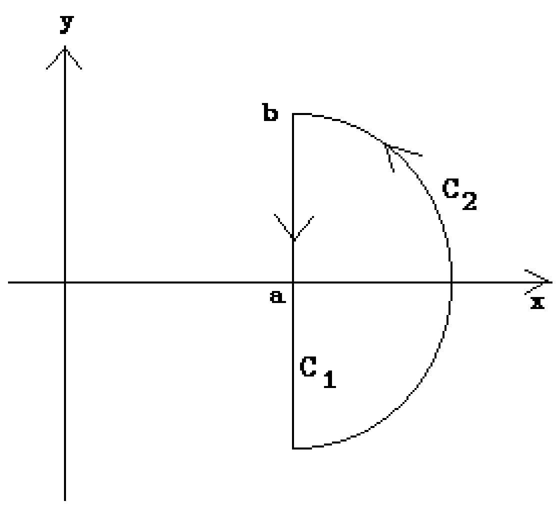
or $$\mathcal{L}[y] = \frac{-s+3}{(s-2)(s+2)}$$

$$\frac{-s+3}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} \quad s = 2 \rightarrow \frac{1}{3} = A, \;\; s = -1 \rightarrow \frac{4}{-3} = B$$

$$L[y] = \frac{\frac{1}{3}}{s-2} + \frac{\frac{-4}{3}}{s+1} \rightarrow \left.\begin{matrix} y=\frac{1}{3}e^{2t}-\frac{4}{3}e^{-t} \\ y'=\frac{2}{3}e^{2t}+\frac{4}{3}e^{-t} \end{matrix}\right\} y' + y = e^{2t}.$$

This method may be extended to higher order equations and to partial differential equations. Note that in order to get $y(t)$ we had to find $\mathcal{L}^{-1}[\phi(s)]$. There exists a general approach to finding $\mathcal{L}^{-1}[\phi(s)]$ which is based on the complex variable theory we have developed.

Let $\phi(z)$ be a function of z analytic on the line $x = a$ and in the entire half plane R to the *right* of this line. Moreover, let $\mid \phi(z) \mid \; \rightarrow \; 0$ uniformly as z becomes infinite through this plane. Let s be any point in the half plane R. We can choose a semicircular contour $C = C_1 \bigcup C_2$ as in the figure below and apply the Cauchy Integral Formula.



$$\phi(s) = \frac{1}{2\pi i} \oint_C \frac{\phi(z)}{z-s} dz = \frac{1}{2\pi i} \int_{a+ib}^{a-ib} \frac{\phi(z)}{z-s} dz + \frac{1}{2\pi i} \int_{C_2} \frac{\phi(z)}{z-s} dz.$$

It can be shown that $\lim_{b \to \infty} \frac{1}{2\pi i} \int_{C_2} \frac{\phi(z)}{z-s} dz = 0$ **so that**

$$\phi(s) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\phi(z)}{s-z} dz$$

Now we shall try to determine the function $f(t)$ whose Laplace Transform is $\phi(s)$.

$$\mathcal{L}^{-1}[\phi(s)] = f(t) = \mathcal{L}^{-1}\left\{\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\phi(z)}{s-z} dz\right\}$$

Hence

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \mathcal{L}^{-1}\left\{\frac{\phi(z)}{s-z}\right\} dz = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \phi(z)\mathcal{L}^{-1}\left\{\frac{1}{s-z}\right\} dz$$

$$= \frac{1}{2\pi i}\int_{a+i\infty}^{a+i\infty} \phi(z)e^{zt} dz.$$

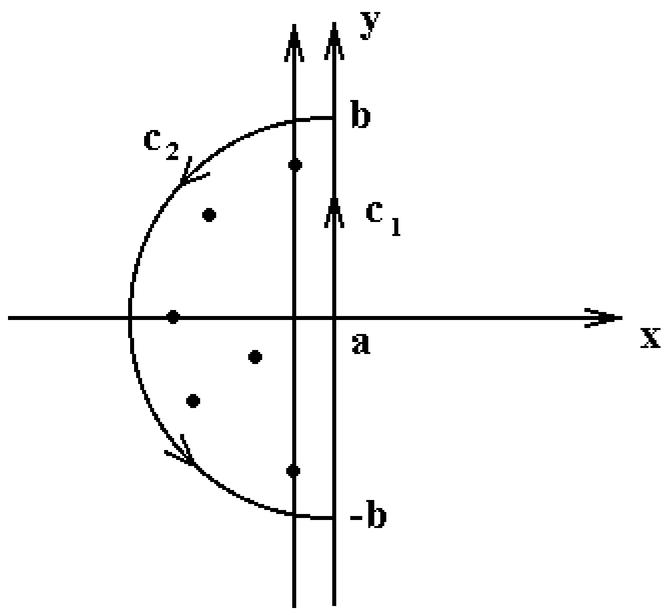
since $\mathcal{L}^{-1}$ only operates on the variable s.

Thus $f(t)$ is given by a line integral in the complex plane taken along a vertical line to the right of all singularities of the transform $\phi(s)$. This complex line integral may be evaluated by the method of residues. A semicircular contour whose diameter is the segment from $a - ib$ to $a + ib$ and whose radius b is large is used. Thus

Theorem: If the Laplace Transform $\phi(s)$ is an analytic function of s except at a finite number of poles each of which lies to the left of the vertical line $Re\ s = a$ and if $s\phi(s)$ is bounded as s becomes infinite through the half plane $Re\ z \leq a$, then

$$\mathcal{L}^{-1}\{\phi(s)\} = \sum \text{residues of } \phi(s)e^{st} \text{ at each of its poles.}$$

Remark. Use



Example: What is $\mathcal{L}^{-1}\left[\frac{1}{(s+a)^2+b^2}\right]$?

Using the theorem we need only compute the residues of $\frac{e^{st}}{(s+a)^2+b^2}$. There are only two first order poles at $-a \pm ib$. At $s = -a + ib$ we have $\lim\limits_{s \to -a+bi}$

$$\lim_{s \to -a+bi} \frac{[s-(-a+ib)]e^{st}}{[s-(-a+ib)][s-(-a-ib)]} = \frac{e^{(-a+ib)t}}{2ib} .$$

At $s = -a - ib$ we have:

$$\lim_{s \to -a-bi} \frac{[s-(-a-ib)]e^{st}}{[s-(-a+ib)][s-(-a-ib]} = \frac{e^{(-a-ib)t}}{-2ib} .$$

Therefore

$$f(t) = \mathcal{L}^{-1}\{\phi(s)\} = \frac{e^{(-a+ib)t}}{2ib} + \frac{e^{(-a-ib)t}}{-2ib} = e^{-at}\left\{\frac{e^{ibt}-e^{-ibt}}{2ib}\right\}$$

$$= \frac{e^{-at} sinbt}{b} .$$