Ma 529
Lecture IV

Linear Algebra

Cramer's Rule

Cramer's Rule Let A be an $n \times n$ matrix, $A = [a_{ij}]_{n \times n}$ and denote by $A_{(j)}$ the $n \times n$ matrix formed by replacing the elements a_{ij} of the jth column of A by the numbers k_i, $i = 1, ..., n$. If $|A| \neq 0$, the system of n linear equations in n unknowns,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= k_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= k_2 \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= k_n
\end{aligned}$$

has the unique solution

$$x_1 = \frac{detA_{(1)}}{detA} \quad x_2 = \frac{detA_{(2)}}{detA}, \cdots x_n = \frac{detA_{(n)}}{detA}$$

Example. **Solve**

$$x + 3y - 2z = 1$$
$$4x - 2y + z = -15$$
$$3x + 4y - z = 3$$

by Cramer's Rule

$$det\ A = \begin{vmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -25$$

$$x = \frac{\begin{vmatrix} 1 & 3 & -2 \\ -15 & -2 & 1 \\ 3 & 4 & -1 \end{vmatrix}}{-25} = -2.8,\ y = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 4 & -15 & 1 \\ 3 & 3 & -1 \end{vmatrix}}{-25} = 1,\ z = \frac{\begin{vmatrix} 1 & 3 & 1 \\ 4 & -1 & -15 \\ 3 & 4 & 3 \end{vmatrix}}{-25} = -9.8$$

Rank of a Matrix: General Linear Systems - App. 1.6, 1.7 P&M

Remark: We cannot use Cramer's Rule if the determinant of the coefficients in a system of n equations in n unknowns equals 0 or if the number of equations $\neq$ number of unknowns.

Ex. **Solve**

$$x_1 + x_2 + 2x_3 + x_4 = 5$$
$$2x_1 + 3x_2 - x_3 - 2x_4 = 2$$
$$4x_1 + 5x_2 + 2x_3 \quad = 7$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 2 & 0 & 7 \end{bmatrix}$$

$$x_1 + x_2 + 2x_3 + x_4 = 5$$
$$0 + x_2 - 5x_3 - 4x_4 = -8$$
$$0 + x_2 - 6x_3 - 4x_4 = -13$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 1 & -6 & -4 & -13 \end{bmatrix}$$

$$x_1 + 0x_2 + 7x_3 + 5x_4 = 13$$
$$0 + x_2 - 5x_3 - 4x_4 = -8$$

$$\begin{bmatrix} 1 & 0 & 7 & 5 & 13 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & -1 & 0 & -5 \end{bmatrix}$$

$$0 + 0 - x_3 = -5$$

$$\begin{bmatrix} 1 & 0 & 0 & 5 & -22 \\ 0 & 1 & 0 & -4 & 17 \\ 0 & 0 & 1 & 0 & 5 \end{bmatrix} \quad \text{Thus } x_3 = 5, x_1 + 5x_4 = -22, x_2 - 4x_4 = 17$$

Definition. A square matrix is said to be <u>nonsingular</u> if its determinant is different from zero; otherwise it is <u>singular</u>.

Ex. $M = \begin{bmatrix} 4 & -5 \\ 3 & 7 \end{bmatrix}$ **det** $M = 43 \neq 0$

therefore M **is nonsingular**

$$Q = \begin{bmatrix} -1 & 5 & 2 \\ 3 & 6 & 1 \\ 2 & -10 & -4 \end{bmatrix} \quad \det Q = 0 \Rightarrow Q \text{ is singular}$$

Definition. The <u>rank</u> of an $m \times n$ **matrix** A **is the largest integer** r **for which a nonsingular** $r \times r$ **submatrix of** A **exists.**

Ex.(1) If $A = [a_{ij}]_{n \times n}$ **is nonsingular then** A **has rank** $r = n$.

(2) Rank of $\begin{bmatrix} 2 & 3 & 0 & -1 \\ 8 & -5 & -6 & 0 \end{bmatrix}$ **is 2 because det** $\begin{bmatrix} 2 & 3 \\ 8 & -5 \end{bmatrix} = -34 \neq 0$

(3) Rank of the 2×3 **matrix**

$\begin{bmatrix} 2 & -3 & 1 \\ -4 & 6 & -2 \end{bmatrix}$ **is 1 since** $\begin{vmatrix} 2 & -3 \\ -4 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -4 & -2 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ 6 & -2 \end{vmatrix}$
$= 0$

Clearly $r \leq$ **smaller of** (m, n) **.**

Elementary Row Operations

Definition. The <u>elementary</u> <u>row</u> <u>operations</u> on an $m \times n$ **matrix are:**
(i) interchanging 2 rows;
(ii) multiplication of a row by a nonzero constant;
(iii) addition to one row of a nonzero multiple of another row, element by element.

Remark. If m or n is large then it is tedious to find the rank of the matrix by evaluating determinants. An alternative is provided by the elementary row operations.

Theorem. If one $m \times n$ matrix can be obtained from another by an elementary row operation, the ranks of the two matrices are equal.

True by the properties of determinants.

Definition. An $m \times n$ matrix is said to be in <u>echelon</u> <u>form</u> if

(i) each of the first r rows has a nonzero element, and all the elements in the last $m -$ r rows are zero $1 \le r \le m$.

(ii) in the first r rows, the first nonzero element is 1;

(iii) in each of the first r rows, the number of zeros that precede the first 1 is less than in the following row.

Example. The matrices

$$\begin{bmatrix} 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 9 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are all in echelon form. Notice that the first has rank 3 and the third has rank 1.

$$\begin{vmatrix} 0 & -3 & 2 \\ 1 & 4 & -5 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$$

Example. None of the matrices below is in echelon form.

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \text{ violates (i)} \qquad \begin{bmatrix} 1 & 0 & 6 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ violates (ii)}$$

$$\begin{bmatrix} 0 & 1 & -3 & -7 \\ 0 & 0 & 1 & 6 \\ 0 & 1 & 5 & -2 \end{bmatrix} \text{ violates (iii)}$$

Theorem. The rank of a matrix in echelon form is the number r of rows containing a nonzero element.

Theorem. Any nonzero matrix can be reduced to echelon form by successive elementary row operations.

Example. Transform

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -3 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & -7 & 8 \end{bmatrix}$$

to echelon form.

Adding $2 \times R_1$ to R_4 and then interchanging R_2 and R_3 yields

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{bmatrix}$$

Adding $(-1) \times R_3$ to R_4 yields

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Adding $2 \times R_1$ to R_2 yields

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & -1 & 0 & 4 & -8 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Multiplying R_1 and R_2 by -1 yields

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 4 \\ 0 & 1 & 0 & -4 & 8 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{echelon form} \Rightarrow \textbf{rank is 3}$$

Cramer's Rule says that a system of n equations in n unknowns having a nonsingular coefficient matrix has one and only one solution. Let us now examine general systems of m linear equations in n unknowns.

Consider the system of m equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdot\cdot\cdot + a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + \cdot\cdot\cdot + a_{2n}x_n &= k_2 \\ \cdot \qquad \cdot \quad \cdot \qquad\qquad &\quad \cdot \\ \cdot \qquad\quad \cdot \quad \cdot \quad & \\ a_{m1}x_1 + a_{m2}x_2 + \cdot\cdot\cdot + a_{mn}x_n &= k_m \end{aligned}$$

The matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & & \cdot \quad \cdot \\ \cdot & & & & \\ a_{m1} & & & & a_{mn} \end{bmatrix}$ **is called the <u>coefficient</u> <u>matrix</u>.**

The matrix $B = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} & k_1 \\ a_{21} & \cdot & \cdot & \cdot & \cdot & k_2 \\ \cdot & \cdot & & & \cdot & \cdot \quad \cdot \\ \cdot & & & & & \\ a_{m1} & \cdot & & & a_{mn} & k_m \end{bmatrix}$

is the called the <u>augmented</u> <u>matrix</u> of the system.

Theorem. A system of m linear equations in n unknowns has a solution if and only if the rank of its coefficient matrix is equal to the rank of its augmented matrix. Moreover, when the coefficient matrix and the augmented matrix have the same rank r :

(1) if $r = n$, the system of linear equations has one and only one solution.

(2) if $r < n$, the system of linear equations has infinitely many solutions.

Example. Find the solutions, if any, of the linear system

$$\begin{aligned} 5x + 7y + 4z &= -2 \\ 3x + y + 3z &= 4 \end{aligned}$$

$12x + 20y + 9z = -10$ **Coefficient matrix is** $A = \begin{bmatrix} 5 & 7 & 4 \\ 3 & 1 & 3 \\ 12 & 20 & 9 \end{bmatrix}$

This matrix is singular because det $A = 0$**. Therefore we cannot use Cramer's rule.**

The augmented matrix is

$$B = \begin{bmatrix} 5 & 7 & 4 & -2 \\ 3 & 1 & 3 & 4 \\ 12 & 20 & 9 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 7/5 & 4/5 & -\frac{2}{5} \\ 0 & 1 & -3/16 & -\frac{13}{8} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x + \frac{7}{5}y + \frac{4}{5}z = -\frac{2}{5} \qquad y - \frac{3}{16}z = -\frac{13}{8}$

$\Rightarrow$ **solutions**

$$x = \frac{-17t+30}{16} \quad y = \frac{3t-26}{16} \quad z = t$$

This system has an infinite number of solutions.

Remark. A system may have no solutions.

Ex.
$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 5 \\ 2x_1 + 3x_2 - x_3 - 2x_4 &= 2 \\ 4x_1 + 5x_2 + 3x_3 \quad &= 7 \end{aligned}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -1 & 2 \\ 4 & 5 & 3 & 0 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 7 & 5 & 13 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

The last row or the echelon matrix says

$0x_1 + 0x_2 + 0x_3 + 0x_4 = -5$ **which is impossible**

Thus the system has no solution.

Note: The coefficient matrix has rank 2 and the augmented matrix has rank 3.

The Inverse of a Matrix

Suppose we want to find the inverse of

$$A = \begin{bmatrix} a_{11} & \cdot & \cdot & \cdot & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix}$$

Then we want a matrix B such that $AB = I$. If b_{i1} are the elements in the first column of B then

$$A \begin{bmatrix} b_{11} \\ \cdot \\ \cdot \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ 0 \end{bmatrix} \Rightarrow \textbf{ we must solve A } X = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ 0 \end{bmatrix}.$$

Form $\left[\begin{array}{cc} & 1 \\ \boldsymbol{A} & 0 \\ & \cdot \\ & 0 \end{array} \right]$ **and row reduce this to echelon form.**

If b_{i2} are elements in the second column of B $\Rightarrow$

$$\boldsymbol{A} \begin{bmatrix} b_{12} \\ \cdot \\ \cdot \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \Rightarrow \textbf{ we must solve } AX = \begin{bmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}\textbf{, therefore form}$$

$\left[\begin{array}{cc} & 0 \\ \boldsymbol{A} & 1 \\ & \cdot \\ & 0 \end{array} \right]$ **and row reduce to echelon form.**

In general we need to solve the n systems

$$AX = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

jth slot

We can solve all these systems at once by forming $[A \mid I]$ and then putting this matrix in echelon form.

Remark. The system will have a unique solution $\Leftrightarrow$ det $A \neq 0$ $\Leftrightarrow$ we can use Cramer's rule.

Theorem. A is nonsingular $\Leftrightarrow$ A is invertible $\Leftrightarrow$ A has rank n $\Leftrightarrow$ A can be row reduced to the <u>identity</u> <u>matrix</u>.

Example. Find A^{-1} for $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. Note that det $A = 5 \neq 0$

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & -5 & 1 & -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & +\frac{4}{5} & -\frac{3}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \quad A^{-1} = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

Eigenvalues and Eigenvectors

Vector Spaces

Definition. A vector space V (or linear space) is a collection of objects together with two operations vector addition (+) and scalar multiplication ($\cdot$) which has the following properties:

(1) For all $\vec{u}, \vec{v} \in V$ there corresponds a unique vector $\vec{u} + \vec{v}$ in V (closure)

(2) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutivity)

(3) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associativity)

(4) There exists a vector $\vec{0} \in V$ such that $\vec{u} + \vec{0} = \vec{u}$ for all $\vec{u} \in V$. (identity)

(5) For each $\vec{u} \in V$ there exists a unique vector $-\vec{u}$ such that $\vec{u} + (-\vec{u}) = \vec{0}$

(6) For every scalar c and for each vector $\vec{u} \in V$ there exists a unique vector $c \cdot \vec{u}$ in V.

(7) For all scalars c and d and all vectors $\vec{u}, \vec{v} \in V$

(i) $c(d \cdot \vec{u}) = (cd) \cdot \vec{u}$

(ii) $1 \cdot \vec{u} = \vec{u}$

(iii) $c \cdot (\vec{u} + \vec{v}) = c \cdot \vec{u} + c \cdot \vec{v}$

(iv) $(c + d) \cdot \vec{u} = c \cdot \vec{u} + d \cdot \vec{u}$

Example. Let V consist of vectors which are points in n dimensional Euclidian space, i.e. $\vec{u} = (u_1, u_2, ..., u_n)$.
Define addition by

$$\vec{u} + \vec{v} = (u_1, ..., u_n) + (v_1, ..., v_n) = (u_1 + v_1, ..., u_n + v_n)$$

and scaler multiplication by

$$c \cdot \vec{u} = c(u_1, ..., u_n) = (cu_1, ..., cu_n).$$

This space is called V_n.

Remark. If we let $A = [a_{ij}]_{n \times n}$ be a matrix of scalars and write vectors in V_n as

$$X = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

then the product $Y = AX$ is also a vector in V_n.

The product $Y = AX$ is called a <u>linear</u> <u>transformation</u> of the vector X. Very often in mathematics one wants to know which vectors, if any, are left unchanged in direction by the transformation. Two nonzero vectors have the same

direction if and only if one is a nonzero scalar multiple of the other. Thus if AX is to have the same direction as X we want

$$AX = \lambda X \quad \lambda \text{ some const.}$$

Thus we want to know which vectors X satisfy

$$AX = \lambda X = \lambda IX \text{ or } (A - \lambda I)X = 0$$

This last equation is equivalent to the system

$$\begin{aligned}
&(a_{11} - \lambda)x_1 + a_{12}x_2 + \cdot\cdot\cdot + a_{1n}x_n = 0 \\
&a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdot\cdot\cdot + a_{2n}x_n = 0 \qquad (1) \\
&\;\cdot \\
&\;\cdot \\
&a_{n1}x_1 + a_{n2}x_2 + \cdot\cdot\cdot \quad + (a_{nn} - \lambda)x_n = 0
\end{aligned}$$

By Cramer's rule if the determinant of the coefficients of the above system is <u>not</u> zero then the only solution to (1) is

$$x_1 = x_2 = \cdot\cdot\cdot = x_n = 0, \qquad \text{the trivial solution.}$$

This implies $X = 0$. Since we want a nontrivial solution (nonzero vectors) we want

det $(A - \lambda I) = 0$, i.e.

$$\begin{vmatrix}
a_{11} - \lambda & a_{12} & \cdot & \cdot & a_{1n} \\
\cdot & a_{22} - \lambda & & & \\
\cdot & & & & \\
\cdot & & & & \\
a_{n1} & \cdot & \cdot & \cdot & a_{nn} - \lambda
\end{vmatrix} = 0$$

Now det $(A - \lambda I) = (-1)^n \lambda^n + \cdot\cdot\cdot$ = polynomial of degree n in λ.

Thus if λ is a root of $p(\lambda) =$ det $(A - \lambda I)$, there will exist a solution $x_1, \cdot\cdot, x_n$ of $(A - \lambda I)X = 0$ such that not all the x_i's are zero. This $\Rightarrow$ that the vector

$$X = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

for this value of λ and these x_i's satisfies $AX = \lambda X$ and hence has the same direction under transformation.

The values of λ such that $\det(A - \lambda I) = 0$ are called eigenvalues. The vector X corresponding to an eigenvalue is called an eigenvector of the matrix A.

Example. Find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}$.

$$\det \quad (A - \lambda I) = \begin{vmatrix} 3-\lambda & -4 \\ -4 & -3-\lambda \end{vmatrix} = -(3-\lambda)(+3+\lambda) - 16$$

$$= -9 + \lambda^2 - 16 = \lambda^2 - 25$$

Therefore $p(\lambda) = \lambda^2 - 25 \quad p(\lambda) = 0 \Rightarrow \lambda = \pm 5$. Thus the eigenvalues are ± 5.
The system $(A - \lambda I)X = 0$ is

$$(3-\lambda)x_1 \quad -4x_2 = 0$$
$$-4x_1 + (-3-\lambda)x_2 = 0$$

If $\lambda = 5 \quad \Rightarrow \quad \left.\begin{matrix} -2x_1 - 4x_2 = 0 \\ -4x_1 - 8x_2 = 0 \end{matrix}\right\} \Rightarrow x_1 + 2x_2 = 0$ or $x_1 = -2x_2$

$\Rightarrow$ eigenvector $(-2t,\ t)$ or $X_1 = \begin{bmatrix} -2t \\ t \end{bmatrix}$

If $\lambda = -5 \Rightarrow \left.\begin{matrix} +8x_1 - 4x_2 = 0 \\ -4x_1 + 2x_2 = 0 \end{matrix}\right\} \Rightarrow \ 2x_1 = x_2$

$\Rightarrow$ eignevector $(t,\ 2t)$ or $X_2 = \begin{bmatrix} t \\ 2t \end{bmatrix}$

Note that $AX_2 = \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix} \left[\begin{smallmatrix} t \\ 2t \end{smallmatrix}\right] = \left[\begin{smallmatrix} -5t \\ -10t \end{smallmatrix}\right]$