Ma 529
Lecture 2

Implicit Functions • Jacobian Determinants.

An equation of the form

$$(1)\ \ f(x, y, z, ...) = 0$$

involving any finite number of variables, where f possesses first partials, can be considered as determing one of the variables, say z, as a function of the others. $\Rightarrow$

$$z = \phi(x, y, ...)$$

in some region about any point where (1) is satisfied and $\frac{\partial f}{\partial z} \neq 0$. In such a case (1) defines z as an <u>implicit</u> function of the other variables. If we consider other variables as independent, we can determine the partial derivative of z with respect to any one of them, without solving <u>explicitly</u> for z, by differentiating (1) partially with respect to that variable. To get $\frac{\partial z}{\partial x}$ we have from (1)

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\ \frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z}\ .$$

Note: denom $\neq$ 0.

If n and k variables are related by n eqs. it is <u>usually</u> possible to consider n of the variables as functions of the remaining k variables. However, this is not <u>always</u> possible. For example, suppose x, y, u, v are related by two equations

$$(2)\ \ f(x, y, u, v) = 0 \ \text{ and } \ g(x, y, u, v) = 0.$$

If these equations determine u and v as differentiable functions of x and y, we may differentiate the system with respect to x and y, considering these two vairables to be independant, $\Rightarrow$

$$(3)\ \ \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\ \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\ \frac{\partial v}{\partial x} = 0 \ \text{ and } \ \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}\ \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\ \frac{\partial v}{\partial x} = 0$$

$$(4)\ \ \frac{\partial f}{\partial y} + \frac{\partial f}{\partial u}\ \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\ \frac{\partial v}{\partial y} = 0 \ \text{ and } \ \frac{\partial g}{\partial y} + \frac{\partial g}{\partial u}\ \frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\ \frac{\partial v}{\partial y} = 0$$

Solving (3) for u_x and v_x and (4) for u_y and v_y $\Rightarrow$

$$u_x = -\frac{\begin{vmatrix} f_x & f_v \\ g_x & g_v \end{vmatrix}}{\Delta} \qquad v_x = -\frac{\begin{vmatrix} f_u & f_x \\ g_u & g_x \end{vmatrix}}{\Delta},$$

$$u_y = -\frac{\begin{vmatrix} f_y & f_v \\ g_y & g_v \end{vmatrix}}{\Delta} \qquad\qquad v_y = -\frac{\begin{vmatrix} f_u & f_y \\ g_u & g_y \end{vmatrix}}{\Delta}$$

where $\Delta = \begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}$.

It must be assumed that $\Delta \neq 0$. Unless $\Delta \neq 0$, the desired partial derivatives cannot exist uniquely, so that u and v cannot be differentiable functions of x and y.
The determinant Δ is known as the Jacobian of f and g with respect to u and v and:

$\frac{\partial(f,g)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}$. This may be extended to more variables. For example,

$\frac{\partial(f,g,h)}{\partial(u,v,w)} = \begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix}$ using our notation, if $\frac{\partial(f,g)}{\partial(u,v)} \neq 0$

(5) $$u_x = -\frac{\frac{\partial(f,g)}{\partial(x,v)}}{\frac{\partial(f,g)}{\partial(u,v)}}\,.$$

In general if $n+k$ variables are related by n equations of the form $f_1 = 0$, $f_2 = 0, \ldots, f_n = 0$ where the functions f_k have continuous first partials, then any set of n variables may be considered as functions of the remaining k variables in some neighborhood of a point where the n equations hold, if the Jacobian of the f's with respect to the n dependent variables is not zero at that point.

Example: $f = x + y + z = 0$
$g = x^2 + y^2 + z^2 + 2xz - 1 = 0.$

Can x and y be considered as functions of z?

Now $\frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 2x+2z & 2y \end{vmatrix} = 2(x+z-y).$

Thus, except on the surface $x+z-y=0$, x and y can be considered as functions of z. That is, z can be taken as the independent variables. When $y = x+z$, the equations

become $2(x+z)^2 = 1$ and $2(x+z) = 0$ and hence incompatible. To see if x and z can be taken as the dependent variables, we calculate the Jacobian:

$\frac{\partial(f,g)}{\partial(x,z)} = \begin{vmatrix} 1 & 1 \\ 2x+2z & 2x+2z \end{vmatrix} = 0$. Since the det is $\equiv 0$ we see x and z cannot be taken as the dependent variables. This is also readily verified directly since both equations involve the combination $x+z$ and only y.

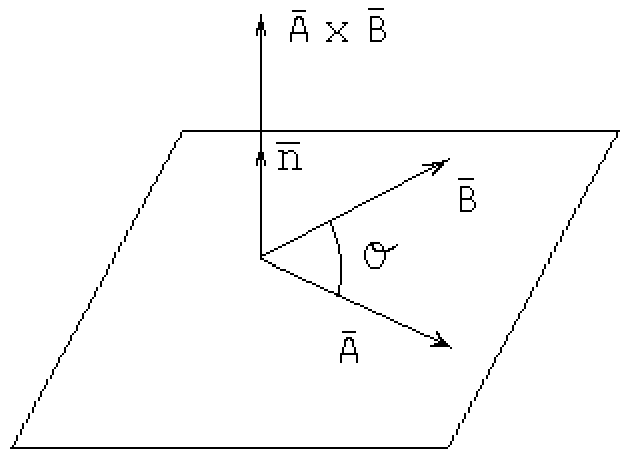
Vector Analysis

Review: Let $\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$; $\vec{B} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ be vectors in x, y, z space. Then

$\vec{A}\cdot\vec{B} = |\vec{A}|\ |\vec{B}|\, cos\theta$ or $\vec{A}\cdot\vec{B} = \sum_{i=1}^{3} a_i b_i$. Here $|\vec{A}| = \left[\sum_{i=1}^{3} a_i^2\right]^{\frac{1}{2}} = (\vec{A}\cdot\vec{A})^{\frac{1}{2}}$

Also

$\vec{A}\times\vec{B} = \vec{n}\,|\vec{A}|\ |\vec{B}|\, sin\theta$. $\vec{n}$ is a unit vector $\perp$ to the plane determined by $\bar{A}$ and $\bar{B}$.



Note: $\vec{A}\times\vec{B} = \vec{0}$ if $\vec{A}$ and $\vec{B}$ are parallel, $\theta = 0°$ or $180°$

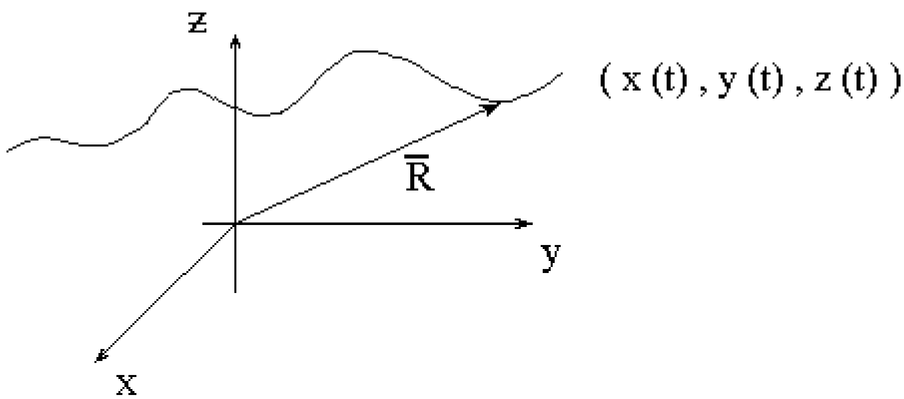
$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Lemma: If $\vec{A}\cdot\vec{B} = 0$ then either $\vec{A} = 0$, $\vec{B} = 0$ or $\vec{A}\perp\vec{B}$. We shall assume the knowledge of cross and dot products. Recall that:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Vector Functions of One Variable

Consider a particle P moving in X, Y, Z space along the curve C. Then $C\colon x(t), y(t), z(t)$.



Let $\vec{R}$ be the vector from the origin to the particle at time t. Then

$$\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}.$$

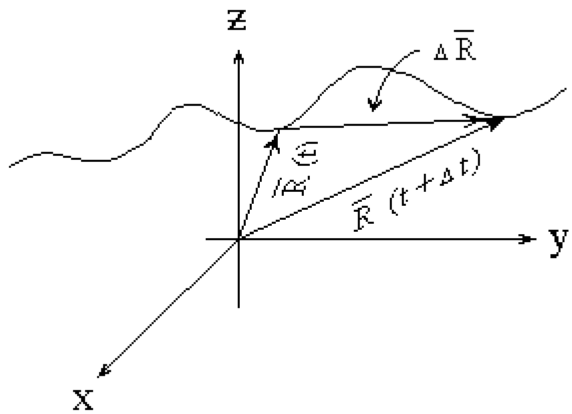
We say $\vec{R}$ is a function of t. In general if $\vec{V} = \vec{V}(t)$ is a function of t, then

$$\vec{V} = \vec{V}(t) = V_1(t)\vec{i} + V_2(t)\vec{j} + V_3(t)\vec{k}.$$

We say $\vec{V}(t)$ is continuous if $V_1, V_2,$ and V_3 are continuous. If $V_1, V_2,$ and V_3 are all differentiable, we define

$$\frac{d\vec{V}}{dt} = \frac{dV_1}{dt}\vec{i} + \frac{dV_2}{dt}\vec{j} + \frac{dV_3}{dt}\vec{k}.$$

Remark: If $\vec{R}$ is the curve vector above, then consider $\vec{R}$ at time t and t $+ \Delta t$.



Note that $\frac{\Delta\vec{R}}{\Delta t}$ has the same direction $\Delta\vec{R}$ since $\Delta t > 0$ is a scalar. As $\Delta t \to 0$ $\frac{\Delta\vec{R}}{\Delta t}$ goes to the direction of the tangent vector.

Example: At what point or points is the tangent to the curve $x = t^3, y = 5t^2, z = 10t$ $\perp$ to the tangent at point when $t = 1$.

C may be represented as $\vec{R}(t) = t^3\vec{i} + 5t^2\vec{j} + 10t\vec{k}$. The tangent to C at any time t is

$$\frac{d\vec{R}}{dt} = 3t^2\vec{i} + 10t\vec{j} + 10\vec{k}.$$

At $t = 1$ the tangent is
$\vec{W} = 3\vec{i} + 10\vec{j} + 10\vec{k}$. To find where $\frac{d\vec{R}}{dt}$ is $\perp$ to $\vec{W}$ we set the dot product equal to 0.

$\frac{d\vec{R}}{dt} \cdot \vec{W} = 9t^2 + 100t + 100 = 0.$ $\Rightarrow t = -\frac{10}{9}, -10$. This yields the points

$\left(\frac{-1000}{729}, \frac{500}{81}, \frac{-100}{9}\right)$ and $(-1000, 5000, -100)$.

If $\vec{U}$ and $\vec{V}$ are two vector functions both differentiable, then it may readily be shown that:

$$\frac{d}{dt}(\vec{U} \cdot \vec{V}) = \frac{d\vec{U}}{dt} \cdot \vec{V} + \vec{U} \cdot \frac{d\vec{V}}{dt}$$

$$\frac{d}{dt}(\vec{U} \times \vec{V}) = \frac{d\vec{U}}{dt} \times \vec{V} + \vec{U} \times \frac{d\vec{V}}{dt}.$$

If $\vec{U} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$, $\vec{V} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, **and** $\vec{W} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$.
Then

$$(*)\frac{d}{dt}[\vec{U}\cdot(\vec{V}\times\vec{W})] = \frac{d\vec{U}}{dt}\cdot\vec{V}\times\vec{W} + \vec{U}\cdot\frac{d\vec{V}}{dt}\times W + \vec{U}\cdot\vec{V}\times\frac{d\vec{W}}{dt}.$$

Since

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad (*)$$

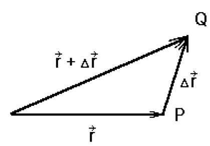
$\Rightarrow$

$$\frac{d}{dt}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} \frac{du_1}{dt} & \frac{du_2}{dt} & \frac{du_3}{dt} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & u_3 \\ \frac{dv_1}{dt} & \frac{dv_2}{dt} & \frac{dv_3}{dy} \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \frac{dw_1}{dt} & \frac{dw_2}{dt} & \frac{dw_3}{dt} \end{vmatrix}$$

This is an interesting identity for 3×3 determinants. It is readily extendable to $n\times n$ determinants.

Vector Operators--The operator ∇

Let $\Phi(x,y,z)$ be a scalar function with first partials Φ_x, Φ_y, Φ_z in some region of x,y,z-space. let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ be the vector drawn from origin to the point $P = (x,y,z)$. Suppose we move from P to a nearby point $Q = (x+\Delta x, y+\Delta y, z+\Delta z)$.



Then Φ will change by an amount $\Delta\Phi$ where

$$\Delta\Phi = \Phi_x \Delta x + \Phi_y \Delta y + \Phi_z \Delta z + \epsilon_1 \Delta x + \epsilon_2 \Delta y + \epsilon_3 \Delta z$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \ \rightarrow \ 0$ as $Q \rightarrow P$. If we divide the change $\Delta\Phi$ by the distance $\Delta s \equiv \mid \Delta\vec{r} \mid$ between P and Q, we obtain a measure of the rate at which ϕ changes when we move from P to Q

$$\frac{\Delta\Phi}{\Delta s} = \phi_x \frac{\Delta x}{\Delta s} + \phi_y \frac{\Delta y}{\Delta s} + \phi_z \frac{\Delta z}{\Delta s} + \epsilon_1 \frac{\Delta x}{\Delta s} + \epsilon_2 \frac{\Delta y}{\Delta s}.$$

Example. If $\Phi(x, y, z)$ is the temperature at the general point $P(x, y, z)$ then $\frac{\Delta\Phi}{\Delta s}$ is the average rate of change in temperature per unit length at the point P in the direction in which Δs is measured.

The limiting value of $\Delta\Phi/\Delta s$ is $Q \rightarrow P$ along the segment PQ is called the derivative of Φ in the direction PQ or simply the <u>directional</u> <u>derivative</u> of Φ. Since $\epsilon_1, \epsilon_2, \epsilon_3 \ \rightarrow \ 0$ as $Q \rightarrow P \ \Rightarrow$

$$(1) \ \frac{d\Phi}{ds} = \frac{d\Phi}{dx} \frac{dx}{ds} + \Phi_y \frac{dy}{ds} + \Phi_z \frac{dz}{ds}.$$

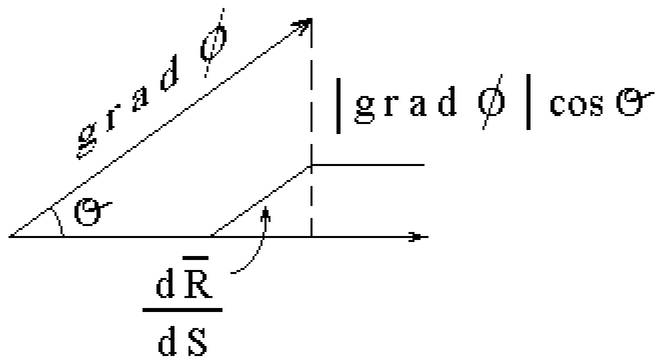
Remark. The first factor in each product in (1) depends only on Φ and the point P. The second factor is independent of Φ and depends only on the direction in which the derivative is being computed. $\Rightarrow$ we write

$$\frac{d\Phi}{ds} = (\Phi_x\vec{i} + \Phi_y\vec{j} + \Phi_z\vec{k}) \cdot \left(\frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j} + \frac{dz}{ds}\vec{k}\right) = (\Phi_x\vec{i} + \Phi_y\vec{j} + \Phi_z\vec{k}) \cdot \frac{d\vec{r}}{ds}.$$

The vector functions $\Phi_x\vec{i} + \Phi_y\vec{j} + \Phi_z\vec{k}$ is known as <u>$gradient$</u> <u>of</u> <u>Φ</u> <u>or</u> $grad\Phi$.

Thus $$\frac{d\Phi}{ds} = (grad\Phi) \cdot \frac{d\vec{r}}{ds}.$$

Remark. The significance of $grad\Phi$. Notice that Δs is length of $\Delta\vec{r} \ \Rightarrow \ \frac{d\vec{r}}{ds}$ is a unit vector. Therefore $grad\Phi \cdot \frac{d\vec{r}}{ds}$ is just the projection of $grad \ \Phi$ in the direction of $\frac{d\vec{r}}{ds} \ \Rightarrow$ $grad\Phi$ has property that its projection in any direction is equal to the derivative of Φ in that direction. Since the maximum projection of a vector is the vector itself, it is clear that $grad\Phi$ extends in the direction of the greatest rate of change of Φ and has that rate of change for its length.



Notation: Often write $grad\Phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\Phi \Rightarrow grad\Phi = \nabla\Phi$ where

$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$. Thus $\frac{d\Phi}{ds} = \nabla\Phi\,\frac{d\vec{r}}{ds}$.

Example. What is the directional derivative of the function $\Phi(x, y, z) = xy^2 + yz^3$ at $(2, -1, 1)$ in the direction of the vector $\vec{i} + 2\vec{j} + 2\vec{k}$?

$$\nabla\Phi = \frac{\partial}{\partial x}(xy^2 + yz^3)\vec{i} + \frac{\partial}{\partial y}(xy^2 + yz^3)\vec{j} + \frac{\partial}{\partial z}(xy^2 + yz^3)\vec{k}$$
$$= y^2\vec{i} + (2xy + z^3)\vec{j} + 3yz^3\vec{k}$$

$$\nabla\Phi\big|_{(2,-1,1)} = \vec{i} - 3\vec{j} - 3\vec{k}.$$

The required directional derivative is the projection of this in the direction of the given vector. We need a unit vector in the direction of the vector $\vec{i} + 2\vec{j} + 2\vec{k}$. $\Rightarrow$

$$\frac{\vec{i} + 2\vec{j} + 2\vec{k}}{\sqrt{1+4+4}} = \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$$

$$\Rightarrow \frac{d\Phi}{ds}\Big|_{(2,-1,1)} =$$
$$\nabla\Phi\Big|_{(2,-1,1)} \cdot \left(\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}\right) = (\vec{i} - 3\vec{j} - 3\vec{k}) \cdot \left(\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}\right) = -\frac{11}{3}.$$

Let us now consider the operator $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$. Given any other vector $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ where $\vec{F} = \vec{F}(x, y, z)$ we can consider $\nabla \cdot \vec{F}$, called the <u>divergence</u> of $\vec{F}$, and $\nabla \times \vec{F}$, called the <u>curl</u> $\vec{F}$

$$\nabla \cdot \vec{F} = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot \left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = div\,\vec{F}$$

$$\nabla \times \vec{F} = curl\,\vec{F} = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \times \left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y}\right)\vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$