Ma 529
Lecture III

<u>Line</u> <u>Integrals</u>

Definition. Let $P(x,y)$ and $Q(x,y)$ be functions of two variables whose first partial derivatives are continuous in an open rectangle H of the x,y - plane. Consider an arc (curve) C in H whose parametric equations are

$$x = f(t) \qquad y = g(t) \qquad a \le t \le b$$

and are such that as t increases from a to b, the corresponding point $(f(t), g(t))$ traces the arc C from the point $A = (f(a),\ g(a))$ to the point $B = (f(b),\ g(b))$. Let f' and g' be continuous for $a \le t \le b$.

Then $\int_C P(x,\ y)\, dx + Q\,(x,y)\, dy$

$$= \int_a^b \Big\{ P(f(t),\ g(t))\ f'(t) + Q\ (f(t),\ g(t)) g'(t) \Big\} dt$$

is called the <u>line</u> <u>integral</u> of $P(x,y)dx + Q(x,y)dy$ along C from A to B.

Remark: Notice that the right hand side above is an ordinary definite integral.

Example: Evaluate the line integral $\int_C (x^2 - y^2)\, dx + 2xy\, dy$

along the curve C whose parametric equations are

$$x = t^2 \ ; \quad y = t^3 \ ; \qquad 0 \le t \le 3/2$$

Solution: $f(t) = t^2$ and $g(t) = t^3$. $\Rightarrow$ $f' = 2t$ and $g' = 3t^2$.

$$\int_C (x^2 - y^2)\, dx + 2xydy = \int_0^{\frac{3}{2}} [\, (t^4 - t^6)\, (2t) + 2t^2\, t^3 (3t^2)\,]\, dt$$

$$= \int_0^{\frac{3}{2}} [\, (2t^5 + 4t^7)\, dt = \tfrac{8505}{512}$$

Remark: C may be described vectorially via

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} \Rightarrow \vec{r}\,'(t) = f'(t)\vec{i} + g'(t)\vec{j}$$

If we let $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$, then

$$\vec{F}(f(t),\ g(t)) = P(f(t), g(t))\vec{i} + Q(f(t), g(t))\vec{j}$$

$$\Rightarrow \vec{F}(f(t), g(t)) \cdot \vec{r}\,'(t) = P(f(t), g(t)) f'(t) + Q(f(t), g(t)) g'(t)$$

Hence

$$\int_C [P(x,y)dx + Q(x,y)dy] \;=\; \int_a^b \vec{F}(f(t),g(t))\,\underbrace{\vec{r}\,'(t)dtdr}_{d\vec{r}} \;=\; \int_C \vec{F}\cdot d\vec{r}$$

Remark: **The results we have given for two dimensions readily go over to three dimensions. We define the 3 dimensional line integral as follows:**

The curve C may be described in three dimensions via

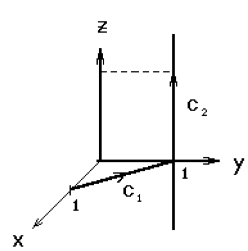
$$x = f(t)\ ;\ \ y = g(t)\ ;\ z = h(t)\ \text{ or }\ \vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$$

If $\vec{F}(x,y,z) = P\,(x,y,z)\,\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$

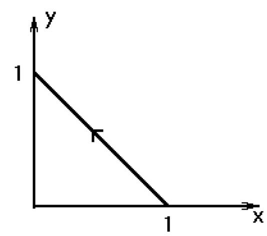
then $\int_C \vec{F}\cdot d\vec{r} \;=\; \int_C Pdx + Qdy + Rdz \;= \int_a^b \vec{F}(f(t),\ g(t),h(t))\cdot \vec{r}\,'(t)dt$

$$= \int_a^b \Big\{ P(f(t),\ g(t),\ h(t))\ f'(t) + Q(f(t),g(t),h(t))g'(t) + R(f(t),g(t),h(t))h'(t) \Big\}dt$$

Example: Compute $\int_C \vec{F}\cdot d\vec{r}$ **where** $\vec{F} = xy\vec{i} + xz\vec{j} - y\vec{k}$ **and** C **is the directed line segment** C_1 **from (1, 0, 0) to (0, 1, 0) followed by** C_2 **which is the segment from (0, 1, 0) to (0, 1, 1).**



lin2.pcx



Solution: On C_1 $z = 0$

$y = -x + 1$ or $x = 1 - y$

Let $y = t \quad x = 1 - t \qquad 0 \le t \le 1$

$$\vec{r}(t) = (1-t)\vec{i} + t\vec{j} + 0 \cdot \vec{k} \;\Rightarrow\; \vec{r}'(t) = -\vec{i} + \vec{j}$$

$$\vec{F} = xy\vec{i} + xz\vec{j} - y\vec{k} = (1-t)\,t\,\vec{i} + 0\vec{j} - t\,\vec{k}$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 [(t - t^2)\vec{i} - t\vec{k}] \cdot [-\vec{i} + \vec{j}]\, dt = \int_0^1 [(t^2 - t)dt] = -\tfrac{1}{6}$$

On C_2 $x = 0$, $y = 1$, z goes from 0 to 1

Let $z = t \quad 0 \le t \le 1 \;\Rightarrow \vec{r}(t) = 0\vec{i} + \vec{j} + t\vec{k}$; $\vec{F} = 0\vec{i} + 0\vec{j} - \vec{k}$ and $\vec{r}'(t) = \vec{k}$

$$\int_{C_2} = \int_0^1 -dt = -1. \;\Rightarrow \int_C = \int_{C_1} + \int_{C_2} = -\tfrac{1}{6} - 1 = -\tfrac{7}{6}$$

Example: Find the value of

$$\int_C y^2 dx + (x - y)dy$$ from the point $A = (0, -2)$ to the point $B = (28, 6)$

(a) along the path $x = t^3 + 1$; $y = 2t$; $-1 \le t \le 3$

(b) along the straight line segment AB

Solution:

(a) first $x = t^3 + 1 \quad y = 2t \rightarrow x = \frac{y^3}{8} + 1$ or $y^3 = 8x - 8$

$$\vec{F}(x,y) = y^2\vec{i} + (x - y)\vec{j} \qquad \vec{r} = (t^3 + 1)\vec{i} + 2t\vec{j}$$

$$\vec{F}(t) = (2t)^2\vec{i} + (t^3 + 1 - 2t)\vec{j} \quad \vec{r}'(t) = 3t^2\vec{i} + 2\vec{j}$$

$$\int_C = \int_{-1}^{3} \left\{ (2t)^2 \cdot (3t^2) + (t^3 - 2t + 1) \cdot 2 \right\} dt = \frac{12t^5}{5} + \frac{2t^4}{4} - \frac{4t^2}{2} + 2t \,\Big|_{-1}^{3}$$

$$= \frac{3088}{5}$$

Along path (b): Line goes from (0, 2) to (28, 6)

$\Rightarrow$ **slope** $m = \frac{6+2}{28} = \frac{2}{7} \Rightarrow y + 2 = \frac{2}{7}x$ **or** $y = \frac{2}{7}x - 2$

Let $x = \frac{7}{2}t \Rightarrow y = t - 2 \quad 0 \leq t \leq 8$

$\vec{F}(t) = (t-2)^2\vec{i} + \left(\frac{7}{2}t - t + 2\right)\vec{j} = (t-2)^2\vec{i} + \left(\frac{5}{2}t + 2\right)\vec{j}$

$\vec{r}(t) = \frac{7}{2}t\vec{i} + (t-2)\vec{j} \Rightarrow \vec{r}\,'(t) = \frac{7}{2}\vec{i} + \vec{j}$

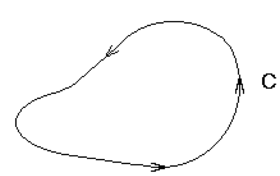
$$\int_C = \int_0^8 \left\{\frac{7}{2}(t-2)^2 + \left(\frac{5}{2}t + 2\right)\right\}dt = \frac{1072}{3}$$

Notice that the two paths give two <u>different</u> results.

<u>Path Independence</u>

Often one must consider situations in which the path C is a closed curve. Hence the starting point A and ending point B are the same. This is usually written as $\oint_C \vec{F} \cdot d\overline{r}$. For plane curves we take the positive direction of C so that the interior of the closed curve is always to the left as C is traversed.

lin3.pcx



Example: Show that $\oint_C \frac{xdy - ydx}{x^2+y^2} = 2\pi$**, where C is the circle** $x^2 + y^2 = a^2$

Solution: Let $x = a\,cos\,t \quad y = a\,sin\,t \quad 0 \leq t \leq 2\pi$

$$\oint_C = \int_0^{2\pi} \left\{\frac{a\,cos\,t\,(a\,cos\,t) - a\,sin\,t\,(-a\,sin\,t)}{a^2}\right\}dt$$

$$= \int_0^{2\pi} \{ cos^2\,t + sin^2\,t\}dt = \int_0^{2\pi} dt = 2\pi$$

We have seen that the value of a line integral depends on the integrand, the endpoints A and B, and the arc C from A to B. However, certain line integrals depend only on the

integrand and endpoints A and B . Such integrals are called <u>path</u> <u>independent</u> or are said to be <u>independent</u> <u>of</u> <u>the</u> <u>path</u>.

Example: Show that the value of the integral

$$\int_C (3x^2 - 6xy)\, dx + (-3x^2 + 4y + 1)\, dy$$

is independent of the path taken from $(-1,\ 2)$ to $(4,\ 3)$.

Solution: Here $P = 3x^2 - 6xy \qquad Q = -3x^2 + 4y + 1$
Suppose we could find a function $G(x, y)$ such that

$$G_x = P \qquad G_y = Q$$

Then $\int_C P\, dx + Q dy = \int_C G_x\, dx + G_y\, dy = \int_C dG = G(4,\ 3) - G(-1,\ 2)$.

This means that we want $P\, dx + Q\, dy$ to be an exact differential. The condition for this is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Here $P_y = -6x = Q_x \Rightarrow$ such a G exists. Now

$$G_x = P = 3x^2 - 6xy$$

$\Rightarrow \quad G = x^3 - 3x^2 y + g(y)$ where $g(y)$ a function of y.
But $G_y = -3x^2 + g'(y) = Q = -3x^2 + 4y + 1 \Rightarrow g'(y) = 4y + 1$ or
$g(y) = 2y^2 + y + C$.

Thus $G(x, y) = x^3 - 3x^2 y + 2y^2 + y + C$ where C a constant.

$$\int_C = G(4,\ 3) - G(-1, 2) =$$
$$64 - 3(16)(3) + 2(9) + 3 + C + 1 + 3(2) - 2(4) - 2 - C = -62$$

We may summarize the above as follows:

Let $P(x, y)\ dx + Q(x, y)\ dy$ be an exact differential of some function G in an open rectangular region H. If C is an arc lying entirely in H with parametric equations

$$x = f(t) \quad y = g(t) \quad t_1 \le t \le t_2$$

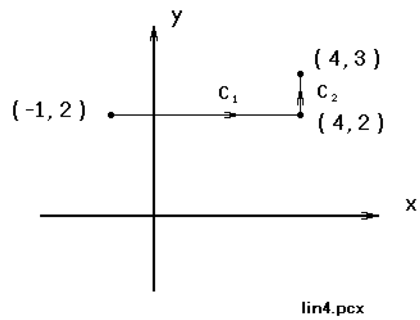
and f' and g' are continuous, then

$$\int_C P(x, y) dx + Q(x, y) dy = G(x_2, y_2) - G(x_1, y_1)$$

where $(x_1, y_1) = (f(t_1),\ g(t_1))$ and $(x_2, y_2) = (f(t_2),\ g(t_2))$ are the endpoints of C.

If a line integral is path independent one may choose a path along which it is easy to evaluate the line integral.

Example: $\int_C \left\{(3x^2 - 6xy)dx + (-3x^2 + 4y + 1)\right\}dy$ from $(-1, 2)$ to $(4, 3)$. (This is the same example we dealt with above.)



$$\int_C = \int_{C_1} + \int_{C_2}$$

Note that $dy = 0$ and $y = 2$ on C_1 and $dx = 0$ and $x = 4$ on C_2 $\Rightarrow$

$$\int_C = \int_{-1}^{4}(3x^2 - 6xy)dx + \int_{2}^{3}(-3x^2 + 4y + 1)dy$$

But $y = 2$ in the first integral whereas $x = 4$ in the second $\Rightarrow$

$$\int_C = \int_{-1}^{4}(3x^2 - 12x)\,dx + \int_{2}^{3}(4y - 47)\,dy = -62$$

Remark: Recall that $\nabla G = G_x\vec{i} + G_y\vec{j}$. Also $d\vec{r} = dx\vec{i} + dy\vec{j}$ $\Rightarrow$ $\nabla G \cdot d\vec{r} = G_x dx + G_y dy$.

Therefore if $Pdx + Qdy$ is an exact differential then

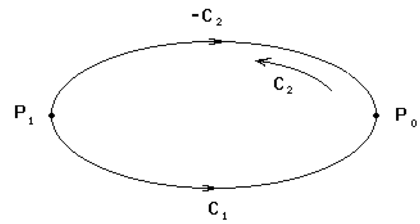
$$\int_C Pdx + Qdy = \int_C \nabla G(x, y) \cdot d\vec{r}$$

Remarks:

(1) The fact that a line integral is independent of path is equivalent to the statement that the line integral around any closed path is zero.

To see this let C be any closed path and $P_o \neq P_1$ be points on C.

lin5.pcx



Then $C = C_1 + C_2$. If the line integral is path independent $\Rightarrow$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{-C_2} \vec{F} \cdot d\vec{r}. \text{ Thus } \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{-C_2} \vec{F} \cdot d\vec{r} = 0$$

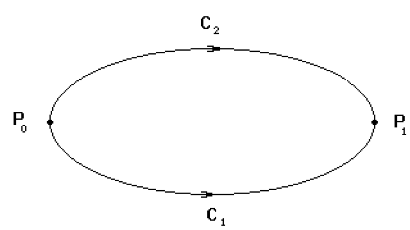
But $-\,d\vec{r}$ along $-C_2$ is equivalent to $d\vec{r}$ along C_2. Therefore

$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot d\vec{r} = 0$$

Suppose now that $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any closed path C.

Let P_0 and P_1 be any two points on C and C_1 and C_2 any two paths joining them.

lin6.pcx



Then $C = C_1 + (-C_2)$ is a closed path and

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1 - C_2} \vec{F} \cdot d\vec{r} = 0$$

$$\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = 0 \text{ or } \int_{C_1} \vec{F} \cdot d\vec{r} = -\int_{-C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

Hence the following are equivalent:

$\int_C \vec{F} \cdot d\vec{r}$ **is path independent** $\leftrightarrow$ **there exists a** G **such that** $\vec{F} = \nabla G$

$\leftrightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$ **for any closed path** C.

We have discussed path independence in two dimensions. Similar things hold in three dimensions.

Example: If $\vec{F} = y\vec{i} - z\vec{j} + x\vec{k}$ **is** $\int_C \vec{F} \cdot d\vec{r}$ **path independent?**

Solution: The line integral is path independent $\Leftrightarrow$ **there exists a function** $\phi(x, y, z)$ **such that** $\nabla\phi = \vec{F}$**. Suppose such a** ϕ **exists.**

$\Rightarrow \phi_x = y; \ \phi_y = -z; \ \phi_z = x$

Now $\phi_z = x \Rightarrow \phi(x, y, z) = xz + g(x, y)$ **But** $\phi_x = z + \frac{\partial g}{\partial x} = y$

$$\Rightarrow z = y - \frac{\partial g}{\partial x}(x, y)$$

But z **is an independent variable and therefore not dependent upon** x **and** y.

Thus no such ϕ **can exist** $\Rightarrow \int_C \vec{F} \cdot d\vec{r}$ **is path dependent for this** $\vec{F}$.

Question: When does there exist a $\phi(x, y, z)$ **such that** $\nabla\phi = \vec{F}$ **?**

Theorem: Suppose $\vec{F}$ **is a continuously differentiable function in a region** D **in space and that**

$$\text{curl}\,\vec{F} = 0 \text{ in } D$$

Then there exists a continuously differentiable, scalar function $\phi(x, y, z)$ **in** D **such that** $\vec{F} = \nabla\phi$.

Remark: $C: \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \quad a \le t \le b$. $\vec{F}$ **force on a particle.**

Then work $= \int_C \vec{F} \cdot d\vec{r}$