Ma 529
Lecture IV

Green's Theorem

There is a remarkable theorem that identifies a double integral over a region R with a line integral around its boundary. It is known as Green's Theorem.

Theorem: Let $P(x,y)$ and $Q(x,y)$ be functions of two variables which are continuous and have continuous first partial derivatives in some rectangular region H in the x, $y-$ plane.

If C is a simple, closed, piecewise smooth curve lying entirely in H, and if R is the bounded region enclosed by C, then

$$\oint_C \left\{ P(x,y)\,dx + Q(x,\ y)dy \right\} = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Corollary: Let R be a bounded region in the $x, y-$ plane. Then the area of R is given by

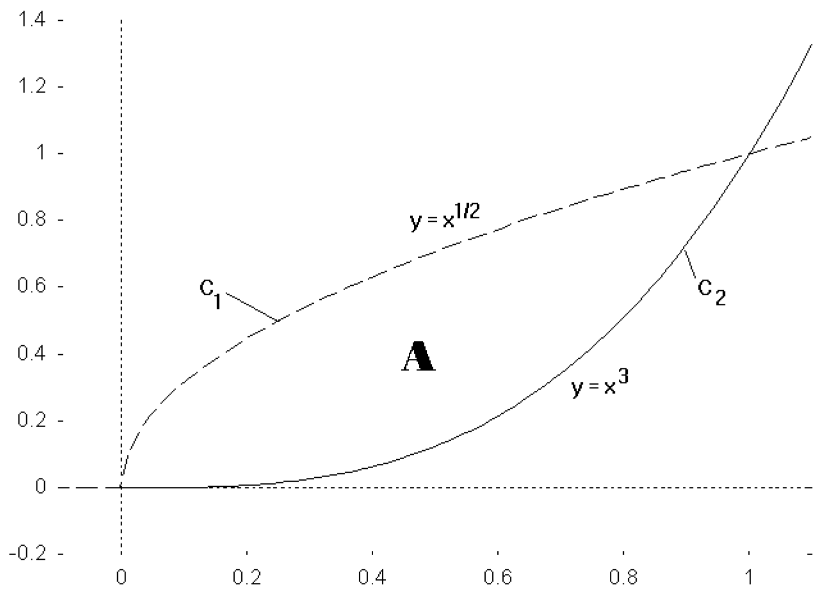
$$A = \tfrac{1}{2} \oint_C (x\,dy - y\,dx)$$

where C is the boundary of R

Proof: Let $P = -y/2$ and $Q = x/2$ in Green's Theorem. $\Rightarrow$

$$\oint_C \left(\tfrac{-y}{2}\,dx + \tfrac{x}{2}\,dy \right) = \int\int_R \left(\tfrac{1}{2} + \tfrac{1}{2} \right) dA = \int\int_R dA = \text{area of } R$$

Example: Find the area of the region A bounded by the curves $y = x^3$ and $y = x^{\frac{1}{2}}$.



Let $C = C_1 + C_2$. Then C is a closed curve which bounds R. We shall use x as the parameter on C and the formula in the corollary. $\Rightarrow$

$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

$$= \frac{1}{2}\int_{C_1}[x(3x^2)\,dx - x^3 dx] + \frac{1}{2}\int_{C_2}\left[x\left(\frac{1}{2}x^{-\frac{1}{2}}\right)dx - x^{\frac{1}{2}}\,dx\right]$$

$$= \frac{1}{2}\int_0^1 2x^3\,dx - \frac{1}{4}\int_1^0 x^{\frac{1}{2}}\,dx = \frac{5}{12}$$

Example: Evaluate the line integral $\oint_C (x^3 + 2y)\,dx + (4x - 3y^2)\,dy$

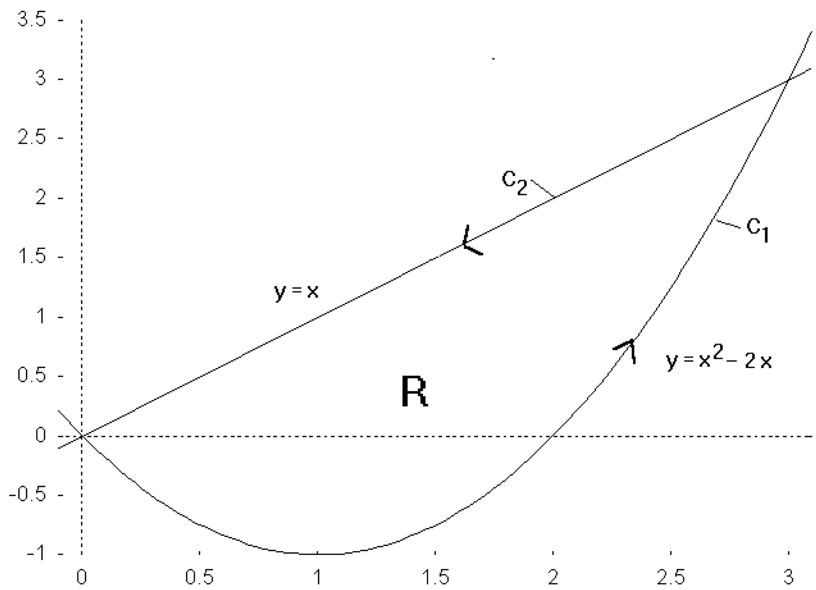
where C is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: $P = x^3 + 2y,\ Q = 4x - 3y^2 \Rightarrow Q_x = 4, P_y = 2$

By Green's Theorem: $\oint_C P\,dx + Q\,dy = \int\int_R (4-2)\,dA = 2\int\int dA$

Therefore we need the area of the ellipse which is $\pi\,ab$. $\Rightarrow \oint = 2\,\pi\,ab$.

Example. Verify Green's theorem for $\oint_C 3xydx + 2x^2dy$ where C is the curve which bounds the region R above by $y = x$ and below by $y = x^2 - 2x$.



Since $P = 3xy$ and $Q = 2x^2$, we see that $Q_x - P_y = 4x - 3x = x$. Thus

$$\int\int (Q_x - P_y)\, dA = \int_0^3 \int_{x^2-2x}^{x} x\, dydx = \tfrac{27}{4}$$

$$C = C_1 + C_2 \qquad C_1\colon\ y = x^2 - 2x \qquad C_2\colon\ y = x \quad 0 \le x \le 3$$

On C_1: $\int_{C_1} Pdx + Qdy = \int_0^3 3x(x^2 - 2x)dx + \int_0^3 2x^2(2x-2)dx$

On C_2: $\int_{C_2} Pdx + Qdy = \int_3^0 3x(x)dx + \int_3^0 2x^2dx$

A straight forward calculation shows that the sum of these last two expressions also equals $\frac{27}{4}$.

Example. Use Green's theorem to evaluate

$$\int_C \left(2y + \sqrt{9 + x^3}\right)\, dx + (5x + e^{\arctan y})dy$$

where C **is the circle** $x^2 + y^2 = 4$. **Now** $Q_x - P_y = 5 - 2 = 3$. $\Rightarrow$

$$\int\int_{x^2+y^2\le 4} 3dA = 3(\text{area of circle of rad 2}) = 3(4\pi) = 12\pi$$

Surface Integrals

There are three common ways of defining a surface:

I. $z = \varphi(x, y)$ (1) as above. Here φ must be a single-valued, continuous function defined on a region of the $x, y -$ plane.

II. Often surfaces are represented by equations of the form $F(x, y, z) = 0$ (2).
If (x_0, y_0, z_0) is a point on such a surface, we can in many cases represent the portion of the surface near (x_0, y_0, z_0) in a form analogous to (1) by solving (2) for $x, y,$ or z in terms of the other two variables.

III. It is frequently convenient to describe a surface by a <u>parametric</u> representation.

Ex. $x = a\ sin\ u\ cos\ v \quad y = a\ sin\ u\ sin\ v. \quad z = a\ cos\ u$

Here u and v are independent parameters. This represents a sphere whose equation is $x^2 + y^2 + z^2 = a^2$. This equation is gotten by elimination of u and v. Note u and v are the spherical coordinates ϕ and θ.

The set of equations

$$x = x(u, v) \quad y = y(u, v) \quad z = z(u, v) \qquad (3)$$

where u and v are parameters, represents an arbitrary surface. This can be seen by eliminating u and v from (3), a procedure that leads to an equation of the form $F(x, y, z) = 0$ which is case II.

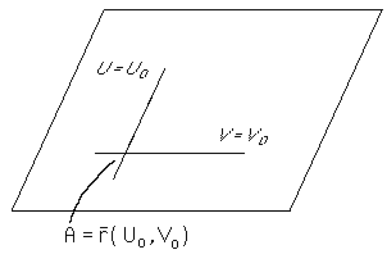
In terms of the radius vector $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ equation (3) for the surface may be written as

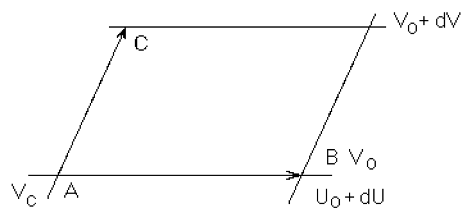
$$\vec{r} = \vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$$

From the parametric equations for a surface it is possible to establish a formula for ds, the element of surface area. In general, ds is obtained by calculating the area between the curves corresponding to:

$$u = u_0,\ u_0 + du,\ v = v_0 \text{ and } v_0 + dv.$$

For infinitesimal areas this element will be essentially planar and have area $ds = |\ \vec{AB} \times \vec{AC}\ |$, where the vectors are the sides of the differential parallelogram shown in the diagram.





$$A = \vec{r}(u_0, v_0) \quad B = \vec{r}(u_0 + du, v_0) = \vec{r}(u_0, v_0) + \tfrac{\partial \vec{r}}{\partial u}(u_0, v_0)du\ + \cdot\ \cdot$$

$$C = \vec{r}(u_0, v_0 + dv) = \vec{r}(u_0, v_0) + \tfrac{\partial \vec{r}}{\partial v}(u_0, v_0)dv + \cdot\ \cdot$$

Thus $\overrightarrow{AB} \ = \frac{\partial \vec{r}}{\partial u}\, du \qquad \overrightarrow{AC} \ = \frac{\partial \vec{r}}{\partial v}\, dv \ \Rightarrow \ ds = \ |\ \frac{\partial \vec{r}}{\partial u} \ \times \ \frac{\partial \vec{r}}{\partial v}\ |\ du\, dv$

Hence, in general, we have for a surface given by

$x = x(u, v) \quad y = y(u, v) \quad z = z(u, v)$ **that**

$\int\int_S f(x, y, z)\, ds = \ \int\int_G f(u, v)\ \ |\vec{r}_u \times \vec{r}_v\ |\ dudv$, **where G is the image of the surface S in the u, v-plane.**

Example. Suppose the surface is given by the representation $z = \varphi(x, y)$ (case I). Let

$x = u, \ \ y = v \ \Rightarrow \ z = \varphi(u{,}v)$ **Then** $\vec{r}(u, v) \ = u\vec{i} \ + v\vec{j} \ + \varphi(u, v)\vec{k}$

also represents the surface. Thus

$$\vec{r}_u = \vec{i} + \varphi_u\vec{k}; \quad \vec{r}_v = \vec{j} + \varphi_v\vec{k}; \qquad \vec{r}_u \times \vec{r}_v = \vec{k} - \varphi_u\vec{i} - \varphi_v\vec{j}$$

$ds = \ |\vec{r}_u \times \vec{r}_v\ |\ du\, dv \ = \ [1 + \varphi_u^2 + \varphi_v^2]^{\frac{1}{2}}\, du\, dv.$ **But since $u = x, \ v = y$ we get**

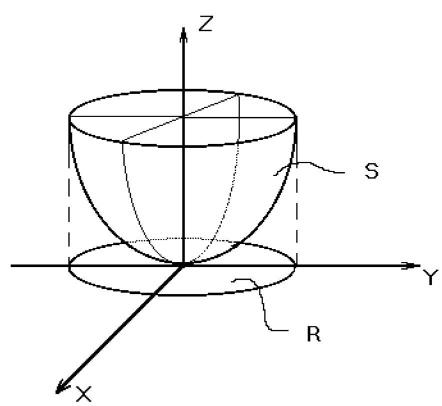
$$ds = [1 + \varphi_x^2 + \varphi_y^2]^{\frac{1}{2}}\, dx\, dy.$$

for a surface given by $z = \varphi(x, y)$.

Example. Find the surface area of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$.

The surface S projects into the interior of the circle $x^2 + y^2 = 1$. This is R. Here $z = \varphi(x, y) = x^2 + y^2$.

Surface area $= \iint_S 1 \cdot dS = \iint_R (1 + \varphi_x^2 + \varphi_y^2)^{\frac{1}{2}}\, dydx$



Here R is circle $x^2 + y^2 \leq 1$. Thus the surface area is given by

$$\iint_R \sqrt{1 + 4x^2 + 4y^2}\, dydx$$

To evaluate this double integral we shall use polar coordinates. Then

Surface area $= \iint_R \sqrt{4r^2 + 1}\, rdrd\theta = \frac{1}{8} \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1}\, r\, drd\theta$

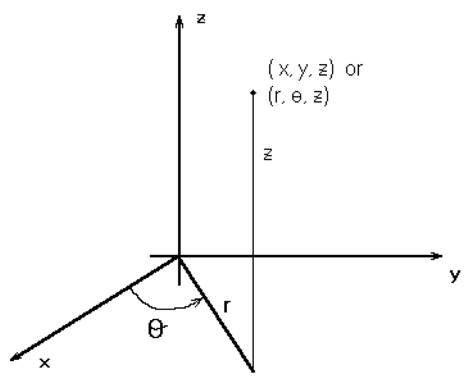
$$= \frac{1}{8} \int_0^{2\pi} \frac{2}{3} (4r^2 + 1)^{\frac{3}{2}} \Big|_0^1 d\theta = \frac{1}{12} \int_0^{2\pi} (5^{\frac{3}{2}} - 1) d\theta = \frac{\pi}{6} (5\sqrt{5} - 1).$$

Cylindrical and Spherical Coordinates

Cylindrical coordinates are related to Cartesian coordinates via

$$x = rcos\theta \quad y = rsin\theta \quad z = z$$

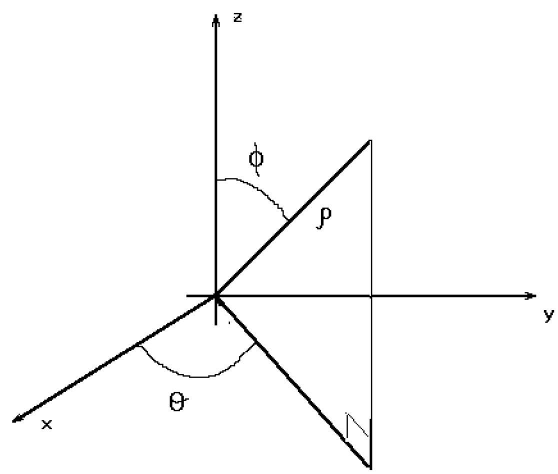
The relationship between a volume element in the two systems is
$dV = dxdydz \rightarrow rdrd\theta dz$, **that is** $\int\int\int dV = \int\int\int rdrd\theta dz$



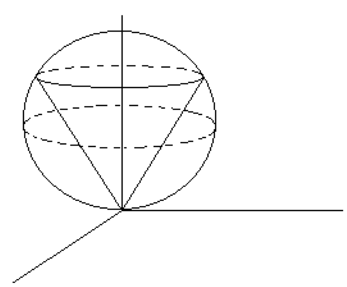
Spherical coordinates are related to Cartesian coordinates via $x, y, z \rightarrow \rho, \theta, \phi$ **where**

$$x = \rho cos\theta sin\phi \quad y = \rho sin\theta sin\phi \quad z = \rho cos\phi \quad 0 \le \theta \le 2\pi \quad 0 \le \phi \le \pi$$

The relationship between a volume element in the two systems is $dV = dxdydz \rightarrow \rho^2 sin\phi d\rho d\theta d\theta$, **that is** $\int\int\int dV = \int\int\int \rho^2 sin\phi d\rho d\theta d\theta$. **It is important to keep in mind that** ϕ **is measured from the** z **axis and thus varies only from** 0 **to** π.



Example. Find the volume above the cone $z^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 2az$.



We shall use spherical coordinates.

Cone: $z^2 = x^2 + y^2$

$z = \rho cos\phi \quad x = \rho cos\theta sin\phi \quad y = \rho sin\theta sin\phi$

The equation of the cone $\Rightarrow \quad \rho^2 cos^2\phi = \rho^2 cos^2\theta sin^2\phi + \rho^2 sin^2\theta sin^2\phi$ **or** $cos^2\phi = sin^2\phi$

$\Rightarrow \; tan\phi = 1 \;\Rightarrow\; \phi = \pm 45° = \frac{\pi}{4}$ **or** $\phi = \frac{\pi}{4} + \pi = \frac{5\phi}{4}$.

Sphere: $x^2 + y^2 + z^2 - 2az = 0$ **or** $x^2 + y^2 + (z-a)^2 = a^2$. **Center at** $(0,0,a)$.

$\Rightarrow \; \rho^2 - 2a\rho cos\phi = 0$ **or** $\rho = 2acos\phi$.

We see that ϕ goes from 0 to $\frac{\pi}{4}$, θ from 0 to 2π. and ρ from 0 to $\rho = 2acos\phi$. Hence

$$\text{Volume } = \int\int\int \rho^2 sin\phi dV_{\rho\theta\phi} = \int_0^{\frac{\pi}{4}} \int_0^{2acos\phi} \int_0^{2\pi} \rho^2 sin\phi d\theta d\rho d\phi$$

Example. Consider a sphere of radius a centered at the origin. We shall find its surface area. Now using spherical coordinates we have

$$x = a\ sin\ u\ cos\ v \quad y = a\ sin\ u\ sin\ v \quad z = a\ cos\ u$$

$$\Rightarrow \quad \vec{r} = a\ sin\ u\ cos\ v\ \vec{i} + a\ sin\ u\ sin\ v\ \vec{j} + a\ cos\ u\ \vec{k}$$

$$\vec{r}_u \times \vec{r}_v = a^2(sin^2\ u\ cos\ v\ \vec{i} + sin^2\ u\ sin\ v\ \vec{j} + sin\ u\ cos\ u\ \vec{k})$$

$$\Rightarrow \quad |\vec{r}_u \times \vec{r}_v| = a^2 sin\ u \qquad ds = a^2 sin\ u\ dudv$$

Thus $\int\int_S ds = \int\int_S a^2\ sin\ u\ dudv = \int_0^{2\pi} \int_0^{\pi} a^2\ sin\ \phi\ d\phi$
$= 4\ \pi\ a^2 =$ surface area of sphere

Surface Elements

Suppose R is a closed rectangular region in the $u, v -$ plane $a \leq u \leq b$, $c \leq v \leq d$. Then the equations $x = x(u,v)$ $y = y(u,v)$, $z = z(u,v)$, where x, y, z are continuous, define a set S which is part of a surface in $x, y, z -$ space. If the functions x, y, z are also 1-1, i.e. distinct points of R are not mapped into the same point of S, then the points of S in $x, y, z -$ space comprise a simple surface element. A simple surface element may be thought of as any configuration which may be obtained from a rectangular plane region by continuous deformation (bending, twisting, stretching, shrinking) without tearing and without bringing together any points which were originally distinct.

If S is a simple surface element corresponding to a rectangular region R in the $u, v -$ plane, the points of S which correspond to the boundary of R form what is called the boundary S. Other points of S are called interior points.

All surfaces may be thought of as being built up out of simple surface elements by matching together portions of the edges of the elements. The boundary of a surface consists of the unmatched edges of its surface elements. If there are no unmatched edges, there is no boundary. For example, a hemisphere has a boundary consisting of its equatorial rim. An entire sphere, an ellipsoid, and the surface of a cube are examples of surfaces without boundary.

A surface is smooth if the functions which parametrize it are continuously differentiable. If a surface is smooth and has no boundary, it is called a smooth surface without boundary. If a surface is given by $F(x, y, z) = 0$, then the surface is smooth without boundary if $\nabla F \neq 0$ for all x, y, z on the surface.

Example. Consider the surface $F(x,y,z) = 4x^2 + 9y^2 - 2z^2 - 8 = 0$. Then $\nabla F = 8x\vec{i} + 18y\vec{j} - 4z\vec{k}$ and $\nabla F = 0 \Rightarrow x = y = z = 0$. But (0,0,0) is not on this surface. $\Rightarrow$ F is smooth without boundary.

Example. Evaluate $\int\int_S f(x,y,z)\, ds$ where $f = x^2$ and S is the part of the cone

$z^2 = x^2 + y^2$ between the planes $z = 1$ and $z = 2$.

We shall use spherical coordinates $x = \rho sin\phi\, cos\theta, y = \rho sin\phi\, sin\theta, z = \rho cos\phi$ In spherical coordinates the equation of the cone is $\phi = \frac{\pi}{4}$. Letting $u = \theta, v = \rho \Rightarrow$ we have for $x, y,$ and z on the surface of the cone that

$$x(u,v) = x(\theta,\rho) = \tfrac{\sqrt{2}}{2}\rho cos\,\theta\,; \quad y(u,v) = y(\theta,\rho) = \tfrac{\sqrt{2}}{2}\rho sin\,\theta\,; \quad z = \tfrac{\sqrt{2}}{2}\rho$$

where $0 \le \theta \le 2\pi$ and $1 \le z \le 2 \Rightarrow \sqrt{2} \le \rho \le 2\sqrt{2}$

$\Rightarrow \vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = \frac{\sqrt{2}}{2}\rho\, cos\,\theta\,\vec{i} + \frac{\sqrt{2}}{2}\rho\, sin\,\theta\,\vec{j} + \frac{\sqrt{2}}{2}\rho\,\vec{k}$

$\Rightarrow \vec{r}_u = \vec{r}_\theta = -\frac{\sqrt{2}}{2}\rho\, sin\,\theta\,\vec{i} + \frac{\sqrt{2}}{2}\rho\, cos\,\theta\,\vec{j}$

$\Rightarrow \vec{r}_v = \vec{r}_\rho = -\frac{\sqrt{2}}{2}\, cos\,\theta\,\vec{i} + \frac{\sqrt{2}}{2}\, sin\,\theta\,\vec{j} + \frac{\sqrt{2}}{2}$

$\Rightarrow \vec{r}_\theta \times \vec{r}_\rho = \frac{1}{2}\rho\,[cos\,\theta\,\vec{i} + sin\,\theta\,\vec{j} - \vec{k}]$ and $|\vec{r}_\theta \times \vec{r}_\rho| = \frac{\sqrt{2}}{2}\rho$

$$\int\int_S x^2\, ds = \int_{\sqrt{2}}^{2\sqrt{2}}\int_0^{2\pi} \tfrac{1}{2}\rho^2 cos^2\,\theta\, \tfrac{\sqrt{2}}{2}\,\rho\, d\theta\, d\rho = \tfrac{\sqrt{2}}{8}\int_{\sqrt{2}}^{2\sqrt{2}} \rho^3\left(\theta + \tfrac{sin 2\theta}{2}\right)\Big|_0^{2\pi} d\rho$$
$$= \tfrac{15}{4}\sqrt{2}\pi$$