Ma 529
Lecture V

Surface Integrals (Continued)

Recall that last time we showed that if S is a surface given parametrically by

$$x = x(u,v),\ y = y(u,v),\ z = z(u,v)$$

then $\int\int_S f(x,y,z)\, ds = \int\int_G f(x(u,v),y(u,v),z(u,v))\ |\vec{r}_u \times \vec{r}_v|\, dudv,$

where G is the image of S in the u,v-plane.

Remark: Very often one is interested in an integral of the form $\int\int_S \vec{v}\cdot\vec{n}\, ds$ **where $\vec{n}$ is a unit normal (perpendicular) vector to the surface S pointing in the outward direction. From the discussion on the top of page 2 it follows that vectors $\vec{r}_u$ and $\vec{r}_v$ are both in the "plane" of the surface. Thus $\vec{r}_u \times \vec{r}_v$ is $\perp$ to the surface S. Hence**

$$\pm\ \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\ \text{ is a unit normal.}$$

We choose the appropriate sign (either + or -) which makes this unit vector outward. One can select an appropriate point on the surface and see if

$$+\ \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\ \text{ is inward or outward.}$$

If it is inward, then use $-\ \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$.

Note that

$$\int\int_S \vec{v}\cdot\vec{n}\, ds\ =\ \int\int_S \vec{v}\cdot\left(\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\right)(\,|\vec{r}_u \times \vec{r}_v|\,)\, du\, dv$$

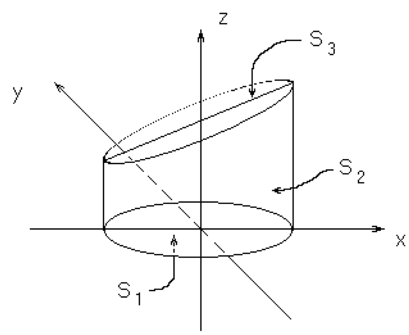
$$=\ \int\int_S \vec{v}\cdot\ (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

Thus, unless one is asked specifically for the unit vector $\vec{n}$, it is not necessary to calculate $|\vec{r}_u \times \vec{r}_v|$.

Example:

Let R be the region bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = x + 2$. Let S be the entire boundary of R. Find the value of $\int\int_S \vec{v} \cdot \vec{n}\, ds$ where $\vec{n}$ is the outward directed unit normal on S and

$$\vec{v} = 2x\vec{i} - 3y\vec{j} + z\vec{k}.$$



Now S is composed of S_1, S_2, and S_3.

On S_1 $\vec{n} = -\vec{k} \Rightarrow \vec{v} \cdot \vec{n} = -z$. But $z = 0$ on $S_1 \Rightarrow \vec{v} \cdot \vec{n} = 0 \quad \Rightarrow \int\int_{S_1} \vec{v} \cdot \vec{n}\, ds = 0$

On S_3 $z = x + 2 \Rightarrow$ we parametrize as $x = u \quad y = v \quad z = u + 2$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = u\vec{i} + v\vec{j} + (u+2)\vec{k}$$

$$\vec{r}_u = \vec{i} + \vec{k} \qquad \vec{r}_v = \vec{j} \qquad \Rightarrow \vec{r}_u \times \vec{r}_v = \vec{k} - \vec{i} \quad \text{so } |\vec{r}_u \times \vec{r}_v| = \sqrt{2}$$

$$\Rightarrow \vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \frac{\vec{k} - \vec{i}}{\sqrt{2}} \qquad \text{and } \vec{v} \cdot \vec{n} = \frac{-2x+z}{\sqrt{2}} = \frac{-2u+u+2}{\sqrt{2}} = \frac{-u+2}{\sqrt{2}}$$

Also $ds = |\vec{r}_u \times \vec{r}_v|\, du\, dv = \sqrt{2}\, du\, dv$ so that

$$\int\int_{S_3} \vec{v} \cdot \vec{n}\, ds = \int\int_G (-u+2) du\, dv$$

Where G is the projection of S_3 in the $u, v-$ plane. But since $u = x$, $v = y$ and the plane $z = x + 2$ slices the cylinder $x^2 + y^2 = 1$, we see that G is the interior of the circle $x^2 + y^2 \leq 1$. Thus on S_3 we have

$$\int\int_{S_3} \vec{v} \cdot \vec{n}\, ds = \int\int_{x^2+y^2 \leq 1} (-x + 2)\, dx\, dy$$

$$= -\int_0^{2\pi}\int_0^1 r\cos\theta\, r\, dr d\theta + 2\iint_{x^2+y^2\le 1} dx\,dy = -\tfrac{1}{3}\int_0^{2\pi}\cos\theta d\theta + 2\pi = 2\pi$$

On S_2 we shall use cylindrical coordinates $x = r\cos\theta \quad y = r\sin\theta \quad z = z$
Since our cylinder is $x^2+y^2=1 \Rightarrow r = 1 \Rightarrow$

$\vec{r} = \cos\theta\,\vec{i}\ \sin\theta\,\vec{j} + z\,\vec{k}$ where $0 \le z \le x + 2 = \cos\theta + 2.$

Taking $u = \theta \quad v = z$ here, we have

$$\vec{r}_\theta = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} \qquad \vec{r}_z = \vec{k}$$

$$\Rightarrow \quad \vec{r}_\theta \times \vec{r}_z = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} \Rightarrow |\vec{r}_\theta \times \vec{r}_z| = 1$$

Thus $\vec{n} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$. This is outward.

$$\vec{v}\cdot\vec{n} = (2\cos\theta\,\vec{i} - 3\sin\theta\,\vec{j} + z\,\vec{k})\cdot\vec{n} = 2\cos^2\theta - 3\sin^2\theta$$

Hence $\iint_{S_2}\vec{v}\cdot\vec{n}\,ds = \int_0^{2\pi}\int_0^{2+\cos\theta}(2\cos^2\theta - 3\sin^2\theta)\,dz d\theta$

$$= \int_0^{2\pi}\int_0^{2+\cos\theta}(2 - 5\sin^2\theta)\,dz\,d\theta = -2\pi$$

Thus we have finally

$$\iint_S \vec{v}\cdot\vec{n}\,ds = \left(\iint_{S_1} + \iint_{S_2} + \iint_{S_3}\right)\vec{v}\cdot\vec{n}\,ds = 0 + 2\pi - 2\pi = 0.$$

Stokes' Theorem and the Divergence Theorem

Stokes' Theorem:

Let S be a regular surface bounded by a closed curve denoted by ∂S (boundary of S). Let $\vec{F}$ and curl $\vec{F}$ be continuous over S.
Then

$$\iint_S \text{curl}\vec{F}\cdot\vec{N}\,ds = \iint_S\left(\vec{\nabla}\times\vec{F}\right)\cdot\vec{N}\,ds = \oint_{\partial S}\vec{F}\cdot d\vec{r}$$

Here the direction of integration around ∂S is positive with respect to the side of S on which the normal $\vec{N}$ is drawn.
Remark:

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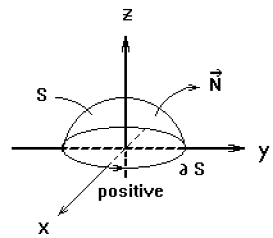

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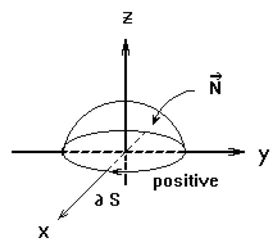


Example: Verify Stokes' Theorem when $\vec{F} = y\vec{i} + 3z\vec{j} + 3x\vec{k}$ and S is the hemispheric surface $z = \sqrt{1 - x^2 - y^2}$.

We shall use the outward $\vec{N}$. We calculate $\oint_{\partial S} \vec{F} \cdot d\vec{r}$ first. Now ∂S is the

circle $x^2 + y^2 = 1, z = 0$. We parametrize this as

$$x = \cos t, y = \sin\ t, z = 0 \quad 0 \le t \le 2\pi$$

$$\vec{F} = \sin t\,\vec{i} + 0\,\vec{j} + 3\cos t\,\vec{k}$$
$$\vec{r}(t) = x\vec{i} + y\vec{j} + z\vec{k} = \cos t\,\vec{i} + \sin t\,\vec{j} + 0\,\vec{k}$$
$$\Rightarrow \vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

Thus $\oint_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -\sin^2 t\, dt = -\pi.$

Now consider $\int\int_S$ curl $\vec{F} \cdot \vec{N} ds$.

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 3z & 3x \end{vmatrix} = -3\vec{i} - 3\vec{j} - \vec{k}$$

S is the surface $x^2+y^2+z^2=1 \;\; z \geq 0$. In spherical coordinates $\rho = 1 \;\; \Rightarrow$

$x = sin\phi \; cos\,\theta, \;\; y = sin\phi \; sin\,\theta, z = cos\,\phi$ Let $u=\phi \quad v=\theta$

and therefore $\vec{r}(u,v) = sinu \; cosv\,\vec{i} \; + \; sin\,u \; sinv\,\vec{j} + cosu\,\vec{k}$

$\vec{r}_u \times \vec{r}_v \; = \; sin^2u \; cosv\,\vec{i} + sin^2\,u \; sinv\,\vec{j} + sin\,u \; cosu\,\vec{k}$

At $\phi = \pi/2, \theta = 0,$ i.e. $u = \pi/2 \quad v = 0 \;\; \Rightarrow$

$\vec{r}_u \times \vec{r}_v = \overline{i}$ which is outward. Hence $\vec{N} = \vec{r}_u \times \vec{r}_v$ is outward

Now curl$\vec{F} \cdot \vec{N} \; = \; -3\,sin^2u \; cos\,v \; - 3sin^2u \; sin\,v \; - sin\,u \; cos\,u$

$\int\int_S$ curl$\vec{F} \cdot \vec{N}ds = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \; (3\,sin^2\,u \; cos \;\; v + 3\,sin^2 \;\; u \; sin\,v + sin \;\; u \; cos\,u)\,du$
dv

$$= -3\int_0^{2\pi}\int_0^{\frac{\pi}{2}} (cos\,v + sin\,v)\,sin^2u\,du\,dv - \int_0^{2\pi}\int_0^{\frac{\pi}{2}} cos\,u\,sin\,u\,du\,dv$$

$$= -\tfrac{3}{2}\int_0^{2\pi} (cos\,v + sin\,v)\left[u - \tfrac{sin\,2u}{2}\right]_0^{\frac{\pi}{2}} dv - \tfrac{1}{2}\int_0^{2\pi} dv$$

$$= -\tfrac{3}{2}\left(\tfrac{\pi}{2}\right)\int_0^{2\pi} [cos\,v + sin\,v]\,dv - \pi$$

$$= -\tfrac{3\pi}{4}\left[-sin\,v + cos\,v\right]_0^{\frac{\pi}{2}} - \pi = -\pi \text{ as before}$$

__The__ __Divergence__ __Theorem__ **(Gauss's Theorem)**

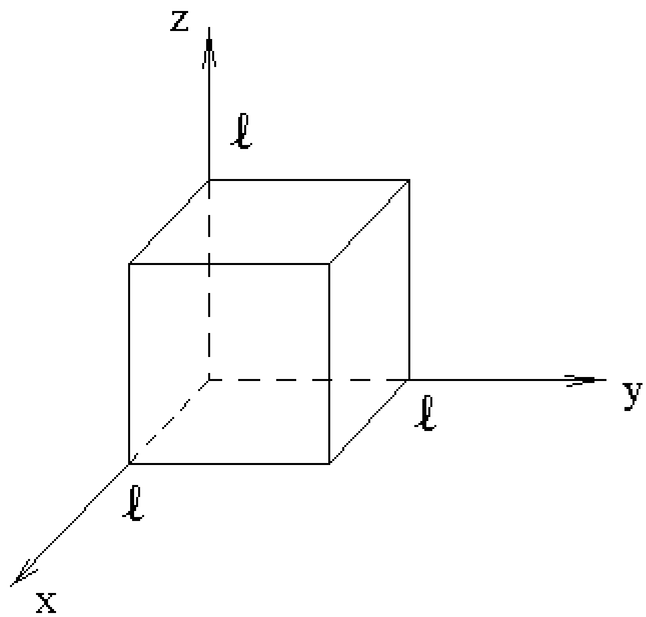
Remark: We shall call a surface __positively__ __oriented__ if the normal $\vec{N}$ is an outer normal; otherwise, S is __negatively__ __oriented__.

Theorem: Suppose S is a regular, positively oriented, closed surface, and that $\vec{F}$ and div $\vec{F}$ are continuous over S and the region V is enclosed by S.

Then $\int\int_S \vec{F} = \int\int_S \vec{F}\cdot\vec{N}\,ds = \int\int\int_V \text{div}\,\vec{F}\,dv = \int\int\int_V \nabla\cdot\vec{F}\,dv$

where $\vec{N}$ is the outward normal to S.
Note: $\vec{N}$ __must__ be outward.

Example: Check the validity of the divergence theorem if $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$, where V is the volume of the cube $0 \;\leq\; x,\; y,\; z \leq \ell$.



div $\vec{F} = 1 + 1 + 1 = 3$. Hence $\int\int\int_V \text{div}\,\vec{F}\,dV = 3\int\int\int_V dV = 3V = 3\ell^3$

Now we must calculate $\int\int_S \vec{F}\cdot\vec{N}ds$ over all six faces of the cube. On $x = \ell$

we use $\vec{N} = \vec{i}$. $\Rightarrow$ $\vec{F}\cdot\vec{N} = (\ell\vec{i} + y\vec{j} + z\vec{k})\cdot\vec{i} = \ell$

$$\underset{\text{Face } x=\ell}{\int\int} \vec{F}\cdot\vec{N}\,ds = \ell \underset{\text{Face } x=\ell}{\int\int} ds = \ell \times \text{(area of face)} = l^3$$

On $x = 0$ $\vec{F} = y\vec{j} + z\vec{k}$ we may take $\vec{N} = -\vec{i}$. Thus $\vec{F}\cdot\vec{N} = 0$ Thus the contribution from this face is 0.

We get similarly for $y = \ell$, $\underset{\text{Face } y=l}{\int\int} \vec{F} = \ell^3$, whereas for $y = 0$, $\int\int \vec{F} = 0$.

And for the face $z = \ell$, $\underset{\text{Face } z=l}{\int\int} \vec{F} = \ell^3$ and on $z = 0$, $\int\int \vec{F} = 0$.

Finally we have $\int\int_S \vec{F}\cdot\vec{N}ds = \ell^3 + \ell^3 + \ell^3 = 3\ell^3$, where S is the entire surface of the cube.

Example. Verify Gauss's Divergence theorem, namely

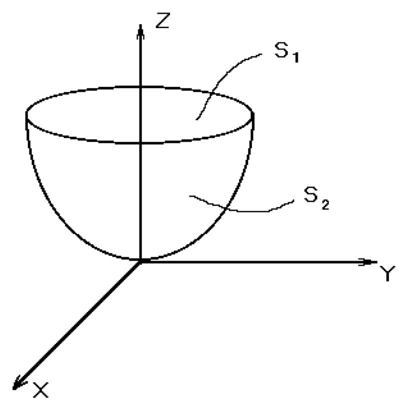
$$\underset{S}{\int\int} \vec{F}\cdot\vec{n}\,ds = \underset{V}{\int\int\int} \text{div}\,\vec{F}\,dv$$

where $\vec{F} = (x - y + z)\,\vec{i} + 2x\vec{j} + \vec{k}$ and S is the closed parabolic bowl consisting of the two pieces

S_2: $z = x^2 + y^2$; $x^2 + y^2 \leq 1$

and

S_1: the circle $x^2 + y^2 \leq 1$, $z = 1$



Thus S_2 is the bowl proper and S_1 is the circular cap on top. Since $\vec{\nabla} \cdot \vec{F} = 1 \Rightarrow$

$$\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dv = \iiint\limits_V 1\, dv = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \int_{x^2+y^2}^{1} dz\, dy\, dx$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx$$

$$= \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta = \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^4}{4}\right) \int_0^1 d\theta = \frac{\pi}{2}$$

We now evaluate $\int\int_S \vec{F} \cdot \vec{n}\, ds = \int\int_{S_1} + \int\int_{S_2}$

On S_2 we use cylindrical coordinates

$$x = r\cos\theta \;,\; y = r\sin\theta \qquad z = z$$

$$\Rightarrow \quad x = r\cos\theta \;,\; y = r\sin\theta \quad z = x^2 + y^2 = r^2$$

Let $r = u\,,\ \theta = v \ \Rightarrow\ x = u\cos v\,,\ y = u\sin v\,, z = u^2\ \ 0 \le u \le 1 \quad 0 \le v \le 2\pi$

$$\Rightarrow \quad \vec{r}(u,v) = u\cos v\,\vec{i} + u\sin v\,\vec{j} + u^2\,\vec{k}$$
$$\vec{r}_u = \cos v\,\vec{i} + \sin v\,\vec{j} + 2u\,\vec{k}$$
$$\vec{r}_v = -u\sin v\,\vec{i} + u\cos v\,\vec{j}$$

$$\vec{r}_u \times \vec{r}_v = \left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \cos v & \sin v & 2u & \cos v & \sin v \\ -u\sin v & u\cos v & 0 & -u\sin v & u\cos v \end{array}\right.$$

$$= -2u^2 \sin v\, j + u\cos^2 v\,\vec{k} + u\sin^2 v\,\vec{k} - 2u^2\cos v\,\vec{i}$$
$$= -2u^2\cos v\,\vec{i} - 2u^2\sin v\,\vec{j} + u\,\vec{k}$$

Note that for $v = \theta = 0\,,\quad r = u = 1$ and we have

$\vec{r}_u \times \vec{r}_v = -2\vec{i} + \vec{k}$ which is inner.

Therefore we use $-\vec{r}_u \times \vec{r}_v = 2u^2\cos v\,\vec{i} + 2u^2\sin v\,\vec{j} - u\vec{k} = \vec{N}$

$$\vec{F} = (u\cos v - u\sin v + u^2)\,\vec{i} + 2u\cos v\,\vec{j} + \vec{k}$$
$$\Rightarrow \vec{F}\cdot\vec{N} = 2u^3\cos^2 v - 2u^3\sin v\cos v + 2u^4\cos v + 4u^3\sin v\cos v - u$$

Therefore $\int\int_{S_2} \vec{F}\cdot\vec{N}\,ds =$

$$\int_0^{2\pi}\int_0^1 [2u^3\cos^2 v + 2u^3\sin v\cos v + 2u^4\cos v - u]\,du\,dv$$

$$= \int_0^{2\pi} [\tfrac{1}{2}\cos^2 v + \tfrac{1}{2}\sin v\cos v + \tfrac{2}{5}\cos v - \tfrac{1}{2}]\,dv$$

$$= \int_0^{2\pi} \{\tfrac{1}{4}(1+\cos 2v)\}dv + [\tfrac{1}{4}\sin^2 v + \tfrac{2}{5}\sin v - \tfrac{1}{2}v]_0^{2\pi}$$

$$= \frac{v}{4} + \frac{\sin 2v}{8}\Big|_0^{2\pi} - \pi$$

$$\int\int_{S_2} \vec{F}\cdot\vec{N}\,ds = \frac{\pi}{2} - \pi = -\frac{\pi}{2}$$

On S_1 : this is the circle $x^2 + y^2 \le 1, z = 1.$ We use the parametization

$$x = r\cos\theta\,,\quad y = r\sin\theta\,,\quad z = 1$$

Therefore $\vec{r}(u,v) = u\,cos\,v\,\vec{i} + u\,sin\,v\vec{j} + \vec{k} \quad 0 \le u \le 1,\ 0 \le v \le 2\pi$

$$\vec{r}_u = cos\,\vec{v} + sin\,v\vec{j} \qquad \vec{r}_v = -u\,sin\,v\vec{i} + u\,cos\,v\vec{j}$$

$$\vec{r}_u \times \vec{r}_v = \left| \begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ cos\,v & sin\,v & 0 & cos\,v & sin\,v \\ -u\,sin\,v & u\,cos\,v & 0 & -u\,sin\,v & u\,cos\,v \end{array} \right.$$

$$= u\,cos^2\,v\,\vec{k} + u\,sin^2\,v\,\vec{k} = u\,\vec{k}.$$

As expected this is outward since $0 \le u \le 1$

Therefore $\vec{N} = u\vec{k}$ and $\vec{F} = (u\,cos\,v - u\,sin\,v + 1)\vec{i} + 2\,u\,cos\,v\,\vec{j} + \vec{k}$

$$\Rightarrow \quad \vec{F} \cdot \vec{N} = u$$

$$\Rightarrow \iint_{S_1} \vec{F} \cdot \vec{N}\,ds = \int_0^{2\pi} \int_0^1 u\,du\,dv = \pi$$

$$\iint_{S_2} + \iint_{S_1} = -\tfrac{\pi}{2} + \pi = \tfrac{\pi}{2}$$

Remark:There are a number of interesting consequences of the divergence theorem. Let $u = u(x,y,z)$ and $v = v(x,y,z)$ be scalar functions with continuous 2nd partials. Also let
$\overline{F} = u \nabla v = uv_x\overline{i} + uv_y\overline{j} + uv_z\overline{k}$. Then
$\nabla \overline{F} = \nabla \cdot (u \nabla v) = u \nabla \nabla v + \nabla u \cdot \nabla v =$
$u \nabla^2 v + u_x v_x + u_y v_y + u_z v_z$. Let $\overline{n}$ be a unit outward normal, i.e., $\overline{n} = \frac{\overline{N}}{|\overline{N}|}$ and apply the divergence theorem to the above $\overline{F} \longrightarrow$

1) $\iiint_v (\nabla u \cdot \nabla v + u \nabla^2 v)dv = \iint_S u[\nabla v \cdot \overline{n}]dS$. Interchange u and v

2) $\iiint_v (\nabla v \cdot \nabla u + v \nabla^2 u)dv = \iint_S v[\nabla u \cdot \overline{n}]dS$. Substract $\longrightarrow$

$$\iiint_v (u \nabla^2 v - v \nabla^2 u)dV = \iint_S (u \nabla v - v \nabla u) \cdot \overline{n}dS.$$

Also known as Green's Theorem.

Remark. **Let us consider the identity (2) in 2-dimensions. Then V becomes a region R and S is the boundary of $R, \partial R$. We have $\int\int_R(\overline{\nabla} u \cdot \overline{\nabla} v + v\overline{\nabla}^2 u)dA = \oint_{\partial R}$ $v\overline{\nabla} u \cdot \overline{n} ds$
where s is arc length along ∂R. Recall $\overline{\nabla} u \cdot \overline{n} = \frac{du}{d\overline{n}}$ = directional derivative of u in direction of outward normal therefore we have**

3) $\int\int_R(\overline{\nabla} u \cdot \overline{\nabla} v + v\overline{\nabla}^2 u)dA = \oint_{\partial R} v \cdot \frac{du}{d\overline{n}}\, ds.$

Let $v = 1 \;\longrightarrow$

4) $\int\int_R \overline{\nabla} u \cdot \overline{\nabla} v + v\overline{\nabla}^2 u)dA \;= \oint_{\partial R} v \cdot \frac{du}{d\overline{n}}\, ds\,.$

Now consider a classical problem in Math Physics. Find u such that $\overline{\nabla}^2 u = 0$ in R and

$\frac{du}{d\overline{n}} = f$ on ∂R. Now (4) $\longrightarrow$ $\int\int_R \nabla^2 u\, dA = 0 = \oint_{\partial R} \frac{du}{d\overline{n}}\, ds = \oint_{\partial R} f\, ds = 0$. The Newmann Problem possesses a solution only if f is such that $\oint_{\partial R} f\, ds = 0$.