Ma 529
Lecture VI

<u>Complex Analysis</u>: Complex numbers and complex functions

Complex numbers: Consider a number of the form $z = x + iy$, where x, y are real and $i^2 = -1$. We call x the *real part* of z and y the *imaginary part*. We write $x = Re(z)$ $y = Im(z)$. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then $z_1 = z_2 \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$. Also

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

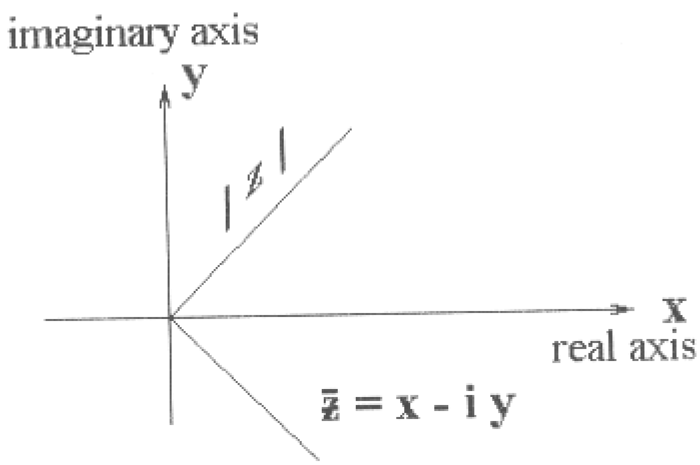
$$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

$z = x + iy = 0 \Leftrightarrow x = 0$ and $y = 0$. If $z_1 \cdot z_2 = 0 \Rightarrow$ at least one of z_1, z_2 is 0.

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2}\left(\frac{x_2-iy_2}{x_2-iy_2}\right) = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + i\left(\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}\right).$$

Remark. If $z = x + iy$, the number $\overline{z} = x - iy$ is called the *complex conjugate* of z. Note that $z \cdot \overline{z} = x^2 + y^2$ which is positive unless $z = 0$.

<u>Geometrical Interpretation of Complex Numbers</u>



Example: Describe the locus of points $z^2 + \overline{z}^2 = 4$.
$z = x + iy \quad \Rightarrow z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$
$\overline{z} = x - iy \quad \Rightarrow \overline{z}^2 = (x - iy)^2 = x^2 - y^2 - 2ixy$

Hence $z^2 + \overline{z}^2 = 2x^2 - 2y^2 = 4$ or $x^2 - y^2 = 2$ which is a hyperbola.

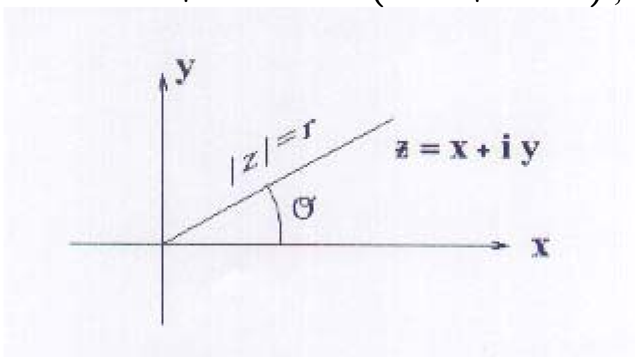
Note that $z\overline{z} = x^2 + y^2 = |z|^2$. This may be used to imply two properties:

(1) $|z_1+z_2| \leq |z_1| + |z_2|$ triangle inequality.

(2) $|z_1-z_2| =$ distance between z_1 and z_2.

Polar Form of Complex Number

Polar coordinates $\Rightarrow z = rcos\theta + irsin\theta = r(cos\theta + i\ sin\theta)$; where



$|z| = r = (x^2+y^2)^{\frac{1}{2}}$ is the distance of point (x, y) from $(0, 0)$ is called the *absolute value* or *modulus* of z. θ is called the *argument* of $z = x + iy$, written $\theta = argz = arg(x+iy)$. θ is determined up to a multiple of 2π. To define $argz$ as a single-valued function of z, one usually either makes the restriction $0 \leq \theta < 2\pi$ or else $-\pi < \theta \leq \pi$. Call this last restriction the principle branch or principle determination. From $x = r\ cos\theta$, $y = r\ sin\theta$ we get $\theta = arctan\ \frac{y}{x}$.

Let $z_1 = r_1(cos\theta_1 + i\ sin\theta_1)$ and $z_2 = r_2(cos\theta_2 + i\ sin\theta_2)$. Then

$z_1z_2 = r_1r_2[cos(\theta_1+\theta_2) + i\ sin(\theta_1+\theta_2)]$

$\Rightarrow |z_1z_2| = r_1r_2 = |z_1|\ |z_2|$ and

$arg(z_1z_2) = \theta_1 + \theta_2 + 2n\pi = argz_1 + argz_2 + 2n\pi$.

Similarly $\left|\frac{z_1}{z_1}\right| = \frac{|z_1|}{|z_2|}$ and $arg\left|\frac{z_1}{z_2}\right| = arg\ z_1 - arg\ z_2 + 2n\pi$ (n any integer). From

$z = r(cos\theta + i\ \ sin\theta)$ we have $z^2 = r^2[cos\ \ 2\theta + i\ \ sin\ \ 2\theta]$ Continuing we have $z^n = r^n[cosn\theta + i\ sin\ n\theta]$.

Theorem: $z^n = r^n(cos\theta + i\ sin\theta)^n$.

Proof: We must show $cos(n\theta) + i\ sin(n\theta) = (cos\theta + i\ sin\theta)^n$. Let $w = cos\theta + i\ \ sin\theta$. Then $|w| = 1 \Rightarrow |w|^2 = |w^2| = 1$ and in general $|w|^n = 1$. Also

$arg\ w^2 = arg(w \cdot w) = arg\ w + arg\ w + 2k\pi = 2\ arg\ w + 2k\pi$.

By induction we have

$$arg(w^n) = n\, arg\, w + 2k\pi.$$

If $arg\ w = \theta$ then $arg(w^n) = n\theta$ (to within $2k\pi$). Therefore the polar form of $w^n = (cos\theta + i\ sin\theta)^n$ is $cos(n\theta) + i\ sin(n\theta)$ and this $\Rightarrow$ the result.

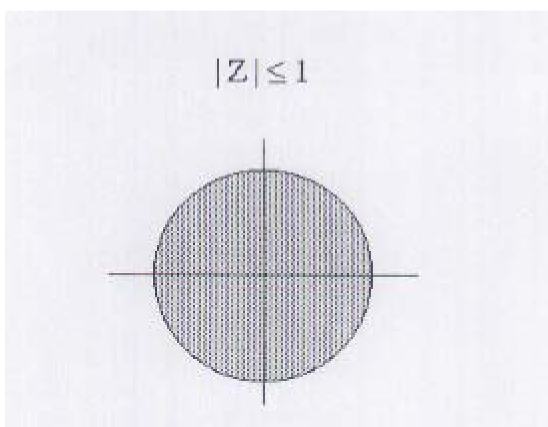
Remark. The result $(cos\theta + i\ sin\theta)^n = cosn\theta + i\ sinn\theta$ is known as de Moivre's Formula. It may be used to derive trig identities.

Functions and Sets in Complex Plane

We shall be discussing functions from the complex numbers to the complex numbers, say $f(z)$. Given a function $f(z)$ we may write $f(z) = Re[f(z)] + iIm[f(z)]$, where $z = x + iy \Rightarrow f(z) = Ref(x+iy) + Imf(x+iy)$. $Re\ f(z)$ and $Im\ f(z)$ are real-valued functions $\Rightarrow f(z) = u(x,y) + iv(x,y)$. Here u and v are functions (real-valued) of the two real variables x and y. We shall need certain properties of complex numbers in our later work. These are:

Definition: Given a complex number $z_0 = x_0 + iy_0$ and a $\delta > 0$, then the open disk of radius δ about z_0 is $\{(x,y) \mid (x-x_0)^2 + (y-y_0)^2 < \delta^2\}$, that is $\{z \mid \mid z - z_o \mid < \delta\}$. An open disk of radius δ about z_0 is called a δ *neighborhood* of z. The closed disk of radius δ about z_0 is $\{z \mid \mid z - z_0 \mid \leq \delta\}$.

Example.



Let S be any set of complex numbers and z any particular complex number. We call members of S *points*.

Definition

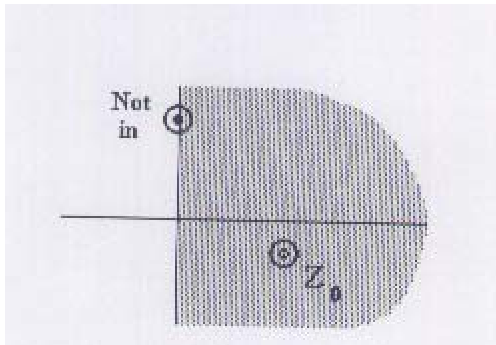
(1) z_0 is an interior point of S if there exists a $\delta > 0$ such that $\{z \mid \mid z - z_o \mid < \delta\} \subseteq S$.

(2) S is <u>open</u> if every point of S is an interior point.

(3) z_0 is a <u>boundary point</u> of S if *every* δ-neighborhood of z_0 contains at least one point in S (z_0 may or may not be in S). The <u>boundary</u> of S consists of all boundary points of S.

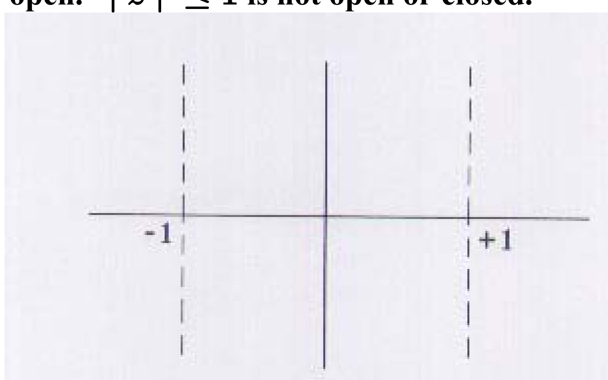
(4) S is <u>closed</u> if every boundary point of S actually belongs to S.

Example. $S = \{z \mid \text{Re} z > 0\} = \{x + iy \mid x > 0\}$.
Every point of S is an interior point. $\Rightarrow$ S is open. Boundary of $S = \{z = iy\}$. S is not closed.



Example. $S = \{1, 2, ..., n\}$ is closed because there are no boundary points of S.

Example. $\mid z \mid\ < 1$ is open. $\mid z \mid\ \leq 1$ is not open or closed.

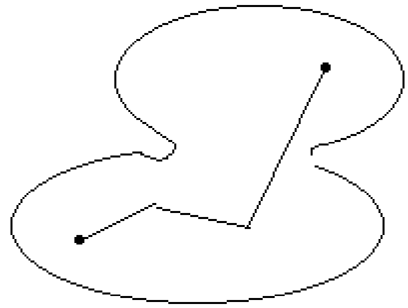


Definition: Let S be a set of complex numbers. Then

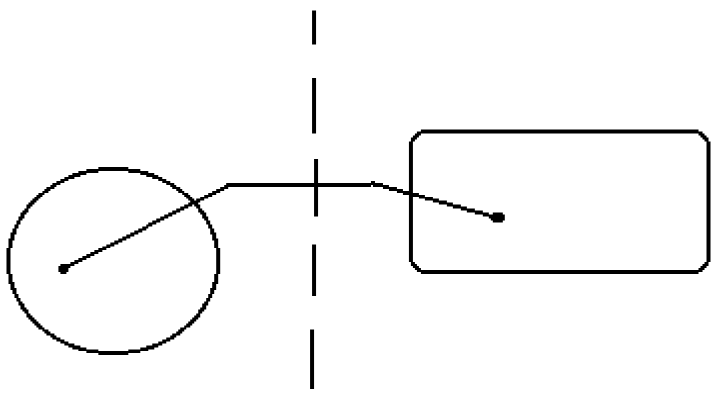
(1) S is *bounded* is $\exists$ a number $M > 0$ such that if $a + bi \in S$, then $a^2 + b^2 < M^2$.
Example: $S = \{1, 2, 3, ...\}$ is not bounded.

(2) S is *connected* if any two points in S can be joined by a polygonal line (a path consisting of finitely straight line segments) consisting only of points of S.

Example.



Connected



not connected

(3) S **is a *domain* if** S **is both open and connected.**

<u>Functions, Continuity, and Differentiability</u>

Let $f(z)$ **be a complex function and** z_0 **a complex number.**

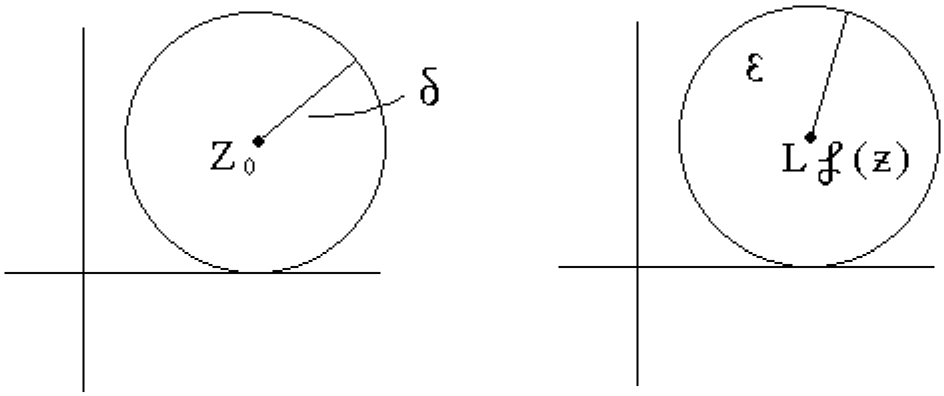
Definition: We say $f(z)$ **has limit** L **as** z **approaches** z_0**, written** $\lim_{z \to z_0} f(z) = L$**. If**

1) $f(z)$ **is defined in some open disk about** z_0**, except possibly at** z_0 **itself.**

2) Given $\in > 0$, $\exists$ **a** $\delta > 0$ **such that** $|f(z) - L| < \in$ **whenever** $|z - z_0| < \delta$.

Note: Independent of direction $z \to z_0$.

Definition: We call $f(z)$ *continuous* at z_0 if $f(z)$ is defined for all z in some open disk about z_0, including z_0, and if $\lim\limits_{z \to z_0} f(z) = f(z_0)$



Definition: $f(z)$ is said to be continuous in the domain D if it is continuous at every point of D.

Definition: A function $w = f(z)$ defined in a domain D is said to be differentiable at a point z of D if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists (is finite) and its value is independent of the way in which $h \to 0$. This limit is denoted by $f'(z)$ or $\frac{dw}{dz}$ and is called the *derivative* of $f(z)$ at the point z.

Example: $f(z) = \mid z \mid$ is continuous, $f'(z)$ does not exist. Consider $\lim\limits_{h \to 0} \frac{|z+h| - |z|}{h}$ (*).
Let $z = r(cos\theta + i\ sin\theta) \Rightarrow \mid z \mid = r$.
a) For $\theta = \theta_0$ constant, let $h = \rho(cos\theta_0 + i\ sin\theta_0)$ (*)

$$\Rightarrow \lim_{\rho \to 0} \frac{|r+\rho| - r}{\rho(cos\theta_0 + i\ sin\theta_0)} = cos\theta_0 - i\ sin\theta_0.$$

b) For $r = r_0$ a constant, take $z + h = r[cos(\theta + \triangle\theta) + isin(\theta + \triangle\theta)]$
Then $\mid z + h \mid = r$ and
(*) $\lim\limits_{\triangle\theta \to 0} \frac{0}{h} = 0 \neq cos\theta_0 - i\ sin\theta_0$.

Hence the derivative does not exist at any point z.

Definition: If a complex function $w = f(z)$ is defined and differentiable at all points of a domain D, we say $f(z)$ is an *analytic* (or regular or holomorphic) function in D.

Remark: The statement "$f(z)$ is an analytic function" means $\exists$ *some* domain of analyticity. Every *polynomial* $f(z) = a_0 + a_1 z + \ldots + a_n z^n$ ($a_0, a_1, \ldots, a_n$ are complex constants) is *regular* in the finite complex plane and every *rational* function of the form

$$\frac{a_0 + a_1 z + \ldots + a_n z^n}{b_0 + b_1 z + \ldots + b_m z^m}$$

(a's and b's are constants) is regular in the finite complex plane, except at the zeroes of the denominator.

Definition: z_0 is called a *singular* point of $f(z)$ if $f(z)$ is *not* differentiable at z_0, but if every neighborhood of z contains points at which $f(z)$ is differentiable.

Example: A zero of the denominator is a singular point of a rational function.

<u>The Cauchy-Riemann Equations</u>

Let $z = x + iy$ and $f(z) = u(x,y) + iv(x,y)$. We want to show that differentiability implies a simple but characteristic property of u and v. Suppose $f'(z)$ exists at a point and let $h = \triangle x + i \triangle y$ and recall that the value of $f'(z)$ is independent of the way in which $h \to 0$. We shall evaluate the limit of the difference quotient:

(1) $\frac{f(z+h) - f(z)}{h} = \frac{[u(x+\triangle x, y+\triangle y) + iv(x+\triangle x, y+\triangle y)] - [u(x,y) + iv(x,y)]}{\triangle x + i\triangle y}$ **by letting**

$h = \triangle x + i \triangle y$ go to zero in two different ways.

1) $\triangle y = 0 \quad \triangle x > 0, \quad \triangle x \to 0$

$$\frac{f(z+h) - f(z)}{h} = \frac{[u(x+\triangle x, y) - u(x,y)]}{\triangle x} + i \, \frac{[v(x+\triangle x, y) - v(x,y)]}{\triangle x} \, .$$

Since limit of the left hand side exists $\Rightarrow$ the limit of right hand side exists and (1) implies

$\lim\limits_{x \to 0} \frac{[u(x+\triangle x, y) - u(x,y)]}{\triangle x}$ **exists and equals $u_x(x,y)$ and (1) also implies**

$\lim\limits_{x \to 0} \frac{[v(x+\triangle x, y) - v(x,y)]}{\triangle x}$ **exists and equals $v_x(x,y)$.**

Therefore at $z = x + iy$

$$f'(z) = u_x + iv_x \qquad \text{(2)}$$

2) $\triangle x = 0, \ \triangle y > 0, \ \triangle y \to 0$. Then (1) $\to$

$$\frac{f(z+h)-f(z)}{h} = \frac{[u(x,y+\triangle y)-u(x,y)]}{i\triangle y} + \frac{i[v(x,y+\triangle y)-v(x,y)]}{i\triangle y} .$$

Let $\triangle y \to 0$, then $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ exist at point z and

$$(3) \qquad f'(z) = v_y - iu_y.$$

From (2) and (3) we have

$$(4) \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} .$$

These are the Cauchy-Riemann equations.

Theorem: A necessary condition for $f(z) = u + iv$ to be differentiable at z is that the four partial derivatives with respect to x and y at $z = x + iy$ are related by equations (4).

Example 1: $f(z) = Re[z] = x$. Here $u = x, \ v = 0$ which implies $u_x = 1, \ v_x = 0$, $u_y = 0, \ v_y = 0$. C-R equations are satisfied nowhere, so that $f(z) = x$ is differentiable nowhere.

Example 2: $f(z) = |z|^2 = x^2 + y^2 \quad u = x^2 + y^2 \quad v = 0$
$u_x = 2x \quad u_y = 2y \quad v_x = v_y = 0$. C.R. equations hold only at $z = 0$. Thus $z = 0$ is only possible point where $|z|^2$ may be differentiable. That this is indeed the case may be determined by a separate computation.

Remark: While C-R equations are necessary for differentiability at a point, they are *not sufficient.*

Example 3: $f(z) = \sqrt{|\, xy \,|} = u \quad v = 0$

Now $f(z) = 0$ on both axes, so that at $z = 0 \quad u_x = v_y = u_y = v_x = 0$. Thus the C-R equations hold at $z = 0$. But $f(z)$ is not differentiable at $z = 0$ since the difference quotient is

(*) $\frac{f(0+h)-f(0)}{h} = \frac{f(h)}{h} = \frac{\sqrt{|\triangle x \triangle| y}}{\triangle x + i\triangle y}$. If we let $\triangle x = \alpha r, \ \triangle y = \beta r$, where α and β real constants,

$\Rightarrow \frac{\sqrt{|\triangle x \triangle y|}}{\triangle x + i\triangle y} = \frac{r\sqrt{|\alpha\beta|}}{r(\alpha+i\beta)} = \frac{\sqrt{|\alpha\beta|}}{\alpha+i\beta}$ and $\lim\limits_{r\to 0} \frac{f(0+h)-f(0)}{h} = \frac{\sqrt{|\alpha\beta|}}{\alpha+i\beta}$. Therefore the difference quotient is not independent of path of approach. Hence $f(z)$ is not differentiable at $z = 0$.