Ma 529
Lecture VII

Theorem: Suppose the four partial derivatives of first order of u and v with respect to x and y exist and are continuous throughout a domain D. Then for $f(z) = u(x,y) + iv(x,y)$ to be regular in D, it is necessary and sufficient that the Cauchy-Riemann equations hold throughout D.

Note: If we are concerned with regularity (or analyticity) at a point, then the existence, continuity of the first partials and Cauchy-Riemann equations must hold in a neighborhood of the point.

Proof: We have done necessity. For sufficiency see O'Neil.

<u>Harmonic Functions</u>: Let us now assume the existence and continuity in D of the second partials of u and v with respect to x, y (which, we will see later, is automatically true). Then Cauchy-Riemann equations $\Rightarrow$ $u_x = v_y$ and $u_y = -v_x$ $\Rightarrow$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ $\qquad$ $\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$ **therefore**

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ $\qquad$ $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Thus u and v both satisfy Laplace's Equation (or the potential equation) in two dimensions of the form $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$.

Definition: A function that has continuous second order partial derivatives in a domain D and satisfies Laplace's equation in D is called a <u>harmonic</u> function.

Remark: Thus the real and imaginary parts of a function analytic in a domain D are harmonic functions in D.

<u>Rational Powers and Roots.</u>

Definition: $z^0 = 1$ $\quad z^n = z^{n-1} \cdot z$ $\quad n = 1, 2, \ldots$. If $z \neq 0$ $\quad z^n = \frac{1}{z^{-n}}$ for $n = -1, -2, \ldots$.

Question: How do we define z^r where r is a rational number?

Answer: r rational $\Rightarrow$ $r = \frac{m}{n}$. We must define $z^{\frac{m}{n}}$, where m and n are integers. Now $z^{\frac{m}{n}} = (z^m)^{\frac{1}{n}}$ $\Rightarrow$ we need a definition of $z^{\frac{1}{n}}$. Let $w = z^{\frac{1}{n}}$ $\Rightarrow$ $z = w^n$. Now Now $z = r[\cos\theta + i\,\sin\theta]$ and $w = R[\cos\phi + i\sin\phi]$ $\Rightarrow$

$$w^n = R^n[\cos\phi + i\sin\phi]^n = R^n[\cos n\phi + i\sin n\phi] = z = r[\cos\theta + i\sin\theta]$$

$\Rightarrow R^n = r$ and $n\phi = \theta + 2k\pi$ (k any integer) $\longrightarrow |W| = R = r^{\frac{1}{n}}$ positive nth root of the positive number r and $arg\, w = \phi = \frac{\theta+2k\pi}{n}$ $k = 0, \pm 1, \ldots$. Thus

$$w = z^{\frac{1}{n}} = r^{\frac{1}{n}} \left[cos\left(\frac{\theta+2k\pi}{n}\right) + i\, sin\left(\frac{\theta+2\pi k}{n}\right)\right].$$

Here k can be an integer. However, we only get distinct values for $k = 0, 1, \ldots, n-1$. For $k = n, n+1, \ldots$ or $k = -1, -2, -3, \ldots$ we repeat these n values. To find $z^{\frac{m}{n}}$ we find $(z^m)^{\frac{1}{n}}$.

Example: Find all possible values of $(2-2i)^{\frac{3}{5}}$.

$(2-2i)^3 = (4-8i-4)(2-2i) = -16-16i$. Therefore we need the fifth root of $-16-16i$. Now $\tan\theta = \frac{y}{x} = \frac{-16}{-16} = +1$. Hence $-16-16i$ is in the third quadrant $\Rightarrow \theta = \pi + \frac{\pi}{4} = \frac{5}{4}\pi$. Thus the fifth roots are:

$$(\sqrt{512})^{\frac{1}{5}} \left[cos\left(\frac{5\pi/4}{5}\right) + i\, sin\left(\frac{5\pi}{20}\right)\right] \qquad k = 0$$

$$(\sqrt{512})^{\frac{1}{5}} \left[cos\left(\frac{(5\pi/4)+2\pi}{5}\right) + i\, sin\left(\frac{5\pi}{20} + \frac{2\pi}{5}\right)\right] \quad k = 1$$

$$= (\sqrt{512})^{\frac{1}{5}} cos\left(\frac{13\pi}{20}\right) + i\, sin\left(\frac{13\pi}{20}\right)$$

$$(\sqrt{512})^{\frac{1}{5}} \left[cos\left(\frac{5\pi}{20} + \frac{4\pi}{5}\right) + i\, sin\left(\frac{21\pi}{20}\right)\right] \qquad k = 2$$

$$(\sqrt{512})^{\frac{1}{5}} \left[cos\left(\frac{5\pi}{20} + \frac{6\pi}{5}\right) + i\, sin\left(\frac{29\pi}{20}\right)\right] \qquad k = 3$$

$$(\sqrt{512})^{\frac{1}{5}} \left[cos\left(\frac{5\pi}{20} + \frac{8\pi}{5}\right) + i\, sin\left(\frac{37\pi}{20}\right)\right] \qquad k = 4$$

Computing the above we get the $\frac{3}{5}$ powers of $2-2i$.

Complex Exponential and Trigonometric Functions

We want to extend real functions to complex functions. We desire a function $f(z)$ such that $f(x) = e^x$ when x replaces z. The fundamental properties of $f(x) = e^x$ are

$f'(x) = f(x)$ and $f(0) = 1$. We shall define e^z analogously. We want an analytic function $f(z)$ which satisfies

(1) $f'(z) = f(z) \quad f(0) = 1$.

If $\exists$ such a function, then it will reduce to e^x for $z = x$. Let $f(z) = u(x,y) + iv(x,y)$. Then (1) $\Rightarrow u_x(x,y) + iv_x(x,y) = u(x,y) + iv(x,y) \quad \Rightarrow \quad u_x(x,y) = u(x,y)$ $v_x(x,y) = v(x,y)$. Solutions to these equations are

(2) $\left.\begin{matrix} u(x,y)=p(y)e^x \\ v(x,y)=q(y)e^x \end{matrix}\right\}$

$f(0) = 1 \Rightarrow u(0,0) + iv(0,0) = 1 \Rightarrow u(0,0) = 1, v(0,0) = 0 \Rightarrow p(0) = 1$ and $q(0) = 0$. The Cauchy Riemann equations $\Rightarrow u_x = p(y)e^x = v_y = q'(y)e^x$ and $u_y = p'(y)e^x = -v_x = -q(y)e^x \Rightarrow p(y) = q'(y)$ and $q(y) = -p'(y)$. Therefore $q(y) = -p'(y) = -q''(y) \Rightarrow p$ and q satisfy the equation.

(3) $\phi'' + \phi = 0$ ODE

Also, $p(0) = q'(0) = 1 \quad q(0) = -p'(0) = 0$. The solutions of (3) are $cosy$ and $siny$. The I.C. $\Rightarrow p(y) = cosy \quad q(y) = siny$, $f(z) = e^x(cosy + i\ siny)$. Therefore we define

(4) $e^z = e^x(cosy + i\ siny)$ where $z = x + iy$

Properties of e^z

(1) $e^z e^w = e^{z+w}$

(2) In (4) set $x = 0,\ y = \theta \Rightarrow e^{i\theta} = cos\theta + i\ sin\theta$ Thus we have a new polar representation for z

$$z = r(cos\theta + i\ sin\theta) = re^{i\theta} = |z|e^{i\theta}$$

(3) $|e^z| = e^x$ and $arg\, e^z = y$

(4) $e^{z+2\pi i} = e^z$ since $z + 2n\pi i = x + i(y + 2n\pi)$ for $n = 0, \pm 1 \pm 2$

(5) $e^{2n\pi i} = 1$

<u>Trigonometric Functions</u>

$$e^z = e^x(cosy + i\ siny).$$

Let $x = 0, y = t \Rightarrow e^{it} = cost + i\ sint$ and therefore

$e^{-it} = cost - i\,sint$. Solving for $cost$ and $sint$ and then setting $t = z \Rightarrow$

(5) $cosz = \frac{e^{iz}+e^{-iz}}{2} \quad sinz = \frac{e^{iz}-e^{-iz}}{2i}$.

We use (5) to define $cosz$ and $sinz$.

Properties :

a) $\frac{d}{dx}\,cosz = -sinz \quad \frac{d}{dz}\,sinz = cosz$

b) $sin(z+w) = sinz\,cosw + cosz\,sinw$

c) $cos(-z) = cosz \quad sin(-z) = -sinz$

d) $cosz = cos(x+iy) = cosx\,cosh\,y - isinx\,sinh\,y$

$sinz = sinx\,cosh\,y + icosx\,sinh\,y$

e) $|\,sinz\,|^2 = sin^2x + sinh^2y \quad |\,cosz\,|^2 = -cos^2x + sinh^2y$

f) $cos^2z + sin^2z = 1$

The Logarithm

The equation $e^w = z \;\; (z \neq 0, \infty)$ has infinitely many solutions.

Definition: Each of the solutions w of $e^w = z \;\; (z \neq 0, \infty)$ is called a *logarithm* of z. The function which associates with each such z, the corresponding values of w, is called the log of z and is denoted by $w = logz$.

Note: The above denotes a multi-valued function. We will use $logz$ to denote any of the "determinations." Let us obtain an explicit representation of $logz$.

Let $w = u + iv$. Then $e^w = e^{u+iv} = e^u \cdot e^{iv} = z \Rightarrow$

$|\,z\,| = e^u \quad v = argz \Rightarrow u = ln\,|\,z\,|$ (natural log) and $v = argz = \theta$.

Hence $logz = ln\,|\,z\,| + i\;argz \;\; (z \neq 0, \infty)$, where $ln\,|\,z\,|$ denotes the real ln and $argz$ is given by all admissible values. Since values of $argz$ differ by multiples of 2π, it follows that the various determinations of $logz$ differ by multiples of $2\pi i$

Definition: The *principal value* or *determination* of the logarithm corresponds to the principal determination of the argument. Thus

$$Lnz = ln \mid z \mid \, + i \; Argz \text{ where } -\pi < Argz \leq \pi.$$

It may be shown that $log(z_1 \cdot z_2) - logz_1 - logz_2 = 0 \pmod{2\pi}$. Also

$$\frac{d}{dz} \, Ln \, z = \frac{1}{z} \quad z \neq 0.$$

Example:
a) Find $log(i^{\frac{1}{4}})$

a) $log(i)^{\frac{1}{4}} = \frac{1}{4} \, logi = \frac{1}{4}[ln \mid i \mid \, + i \; argi]$
$= \frac{1}{4}[ln \mid i \mid \, + i(\frac{\pi}{2} + 2n\pi)] = i(\frac{\pi}{8} + \frac{n\pi}{2}) \;\; n = 0, \pm 1, \ldots$

b) Find $Ln(1 + i\sqrt{3})$

b) $Ln(1 + i\sqrt{3})$
$Ln(1 + i\sqrt{3}) = ln \mid 1 + i\sqrt{3} \mid \, + i \; Arg(1 + i\sqrt{3})$
$= ln \, 2 + i \; Arg(1 + i\sqrt{3})$

Let $\theta = arg(1 + i\sqrt{3}\,)$
$tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{1} = \sqrt{3} \;\Rightarrow\; \theta = \frac{\pi}{3} + 2n\pi \;\Rightarrow\; Arg(1 + i\sqrt{3}) = \frac{\pi}{3}$ Thus $Ln(1 + i\sqrt{3}) = ln \, 2 + i \, \frac{\pi}{3}$

The function z^{α}

If α is an arbitrary complex constant, z^{α} is defined by $z^{\alpha} = e^{\alpha \, logz}$ $(z \neq 0, \infty)$. In general z^{α} is multiple-valued. If we use Lnz instead of $logz$ in the definition of z^{α}, we get a single-valued function called the *principal value* of z^{α}, denoted by $Pr[z^{\alpha}]$.

$$Pr[z^{\alpha}] = e^{\alpha \, Lnz}.$$

Remark. $(z^{\alpha})^{\beta} = z^{\alpha\beta} = (z^{\beta})^{\alpha}$ provided proper determinations are used. (Can use some determination throughout.)

Theorem. z^{α} is an n-valued function (n a positive integer) $\leftrightarrow$ α is a real rational number of the form m/n, where m and n have no common factor.

Example. Find $Pr[(-i)^{i}]$

$$Pr[(-i)^{i}] = e^{iLn(-i)} = e^{iLn(e^{\frac{-\pi i}{2}})} = e^{i(-\pi i/2)} = e^{\frac{\pi}{2}} \; .$$