Ma 529
Lecture IX

Complex Integration

Definition: A continuous curve C in the z-plane is a set of points z for which $z = \phi(t) + i\psi(t)$ (1) $a \le t \le b$ where $\phi(t)$ and $\psi(t)$ are continuous functions in $a \le t \le b$. It is simple if it is non-self-intersecting, i.e., if $\phi(t) = \phi(t')$ where $a < t \le t' < b$, then $t = t'$. It is a simple closed curve (a Jordan curve) if $\phi(t) = \phi(t')$ and $\psi(t) = \phi(t')$ implies $t = a, t' = b$.

Definition: The continuous curve (1) is said to be a smooth arc if $\phi(t)$ and $\psi(t)$ have continuous derivatives and $[\phi'(t)]^2 + [\psi'(t)]^2 \neq 0$ in $a \le t \le b$.

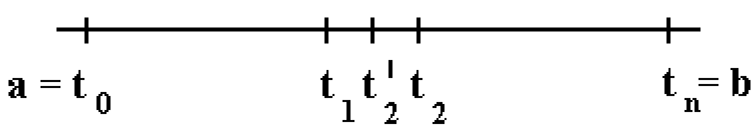
Note: A curve C is piecewise smooth if C consist of finitely many smooth pieces. This happens when $[a, b]$ can be broken up into finitely many intervals, $[a, t_1], [t_1, t_2]...[t_{n-1}, b]$.

$z(t) = \phi(t) + i\psi(t)$ is smooth on each subinterval.

The Complex Line Integral

Let C be a continuous curve in the z-plane defined by $z = \phi(t) + i\psi(t)$ $a \le t \le b$. Let $f(z) = u(x, y) + iv(x, y)$ can be defined on C.

From the subdivision



$$t_0 < t_1 < ... < t_n$$

$\triangle : a = t_0, t_1, ..., t_n = b$; let $z_j = z(t_j)$. Consider the sum $\displaystyle\sum_{j=1}^{n} f(w_j)(z_j - z_{j-1})$

where $w_j = z(t'_j)$. $(t_{j-1} \le t'_j \le t_j)$ is on C. Let $|\, z_j - z_{j-1} \,| \rightarrow 0$ as $n \rightarrow \infty$, i.e., $\| \Delta \| \rightarrow 0$.

Definition: If the limit $\rightarrow \displaystyle\lim_{\| \Delta \| \rightarrow 0} \sum_{j=1}^{n} f(w_j)(z_j - z_{j-1})$ exists, it is called the line

integral of $f(z)$ along C and is denoted by $\int_C f(z)dz$.

Theorem: If C is a smooth arc and $f(z)$ is continuous along C, then $\int_C f(z)dz$ exists.

Indeed $\int_C f(z)dz = \int_C (u+iv)(dx+idy)$

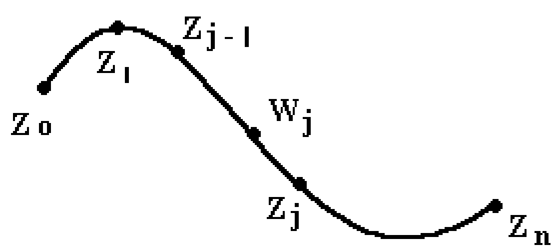
$$= \int_a^b [u(\phi(t),\psi(t)\Big) + iv(\phi(t),\psi(t)\Big)][\phi'(t)+i\psi'(t)]dt.$$

Remark: We may restate the above as: $\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt$

$z = z(t), a \leq t \leq b.$

Properties of the Complex Line Integral

1) $\int_C [f(z)+g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$



2) **If k is a constant, then** $\int_C kf(z)dz = k\int_C f(z)dz$

3) **If C' denotes C described in the opposite direction, then** $\int_{C'} f(z)dz = -\int_C f(z)dz$

Definition: A contour (or piece-smooth curve) is a continuous curve consisting of a finite number of smooth arcs.

4) **If the contour C is made up of the smooth arcs $C_1, C_2, ..., C_n$; then**

$$\int_C f(z)dz \equiv \left\langle \int_{C_1} + \int_{C_2} + ... + \int_{C_n} \right\rangle f(z)dz$$

5) If M is an upper bound of $|f(z)|$ on the smooth arc C (where $f(z)$ is continuous on C), i.e. $|f(z)| \leq M\ \forall z$ on C, and L is the length of C, then $\left|\int_C f(z)dz\right| \leq ML$.

Example 1: $f(z) = K$ a constant. C is a continuous curve from $z = \alpha$ to $z = \beta$. Then $(z_0 = \alpha, z_n = \beta)$.

Figure

$\sum_{j=1}^{n} f(w_j)(z_j - z_{j-1}) = K \sum_{j=1}^{n}(z_j - z_{j-1}) = K(z_n - z_0) = K(\beta - \alpha)$ therefore

$$\int_C K dz = K(\beta - \alpha) = \int_\alpha^\beta K dz \text{ independent of path.}$$

Example 2: $f(z) = z$; C is any piece-smooth curve from $z = \alpha$ to $z = \beta$.

$\int_C z dz = \frac{1}{2}(\beta^2 - \alpha^2) = \int_\alpha^\beta z dz$ This is independent of path. See O'Neil for details.

Example 3: Calculate $\int_C \frac{dz}{z}$ where C is a circle of radius ζ centered at the origin.

Let $x = \zeta cost = \phi(t) \quad y = \zeta sint = \psi(t)$; $0 \leq t \leq 2$
$z(t) = \zeta(cost + isint) = \zeta e^{it}$
$z'(t) = \phi'(t) + i\psi'(t) = \zeta i e^{it} \rightarrow dz = z'(t)dt = \zeta i e^{it}dt \rightarrow$
$\int_C \frac{dz}{z} = \int_0^{2\pi} \frac{z'(t)dt}{z(t)} = \int_0^{2\pi} \frac{\zeta i e^{it}}{\zeta e^{it}}\, dt = \int_0^{2\pi} i dt = 2\pi i.$

Note: If we know a function $F(z)$ is regular and singled valued in a domain D such that $F'(z) = f(z) \quad C\colon z_0 \rightarrow z_1$.

Then $\int_C f(z)dz = \int_{z_0}^{z_1} f(z)dz = \int_a^b F'(z)\frac{dz}{dt}dt$
$= F[z(b)] - F[z(a)] = F(z_1) - F(z_0).$

Example: $\int_C z^n dz$; C is closed contour; $n \geq 0$, n any integer.

$$z = F'(z) \text{ where } F(z) = \frac{z^{n+1}}{n+1}$$

$\int_{z_0}^{z_1} z^n dz = F(z_1) - F(z_0)$. Here $z_1 = z_0 \rightarrow \int_C z^n dz = 0 \quad C$: closed contour.

The Cauchy Integral Theorem

Terminology: ***path*** **- piecewise-smooth curve;** ***closed path*** **- terminal and initial points coincide;** ***simple*** **- does not cross itself. A domain D is** ***simply connected*** **if every simple closed curve in D encloses only points of D.**

Example: $|z| < 1$ simply connected; $0 < |z| < 1$ not simply connected.

<u>Cauchy Integral Theorem</u>. Let $F(z)$ be analytic in a simply connected domain D. Then for every simple closed path C in D $\int_C f(z)dz = 0$.

Convention: If C is a simple closed continuous curve the point z is said to describe C in the ***positive sense*** **if, as it moves along, the points of this interior domain in the immediate vicinity lie to its left.**
Example: In order to understand the hypothesis of C.I.T. we give some examples:

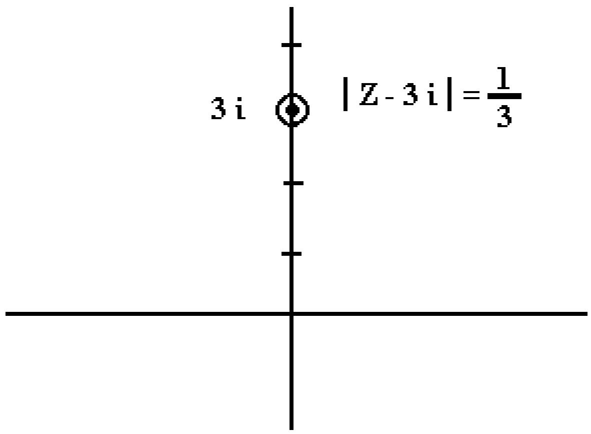
1) We showed before $\int_C z^n dz = 0$ for any nonnegative integer n and any simple closed path in the plane. By C.I.T. this is true because z^n is analytic everywhere and the entire plane is a simply connected domain. $\left(\text{Note: } \frac{d}{dz}\ \frac{z^{n+1}}{n+1} = z^n\right)$.

2) Consider $\int_C \frac{dz}{z}$ where C is any circle of radius ζ about origin.

Showed: $\int_C \frac{dz}{z} = 2\pi i$. We do not get zero even though $\frac{1}{z}$ is analytic on C. Cauchy I.T. does not hold because $\frac{1}{z}$ is not analytic at 0 which is inside the region bounded by C. There does not exist a simply connected domain D containing C in which $\frac{1}{z}$ is analytic.

3) In general, any time C goes around one or more points where $f(z)$ is not analytic, C.I.T. fails to hold. In such a case $\int_C f(z)dz = 0$ may or may not hold. Theorem does not hold. We have seen that $\int_C z^n dz = 0$ for $n < -1$ even though C.I.T. does not hold.

Example. $\oint_C \frac{(2z+1)}{z^3 - iz^2 + 6z} dz$. C is the circle of radius $\frac{1}{3}$, about $3i$.



$$\underbrace{\frac{z+1}{z^3-iz^2+6z}} = \frac{A}{z} + \frac{B}{z+2i} + \frac{C}{z-3i}$$

$$\underbrace{\underline{z(z^2-iz+6)}}_{z(z+2i)(z-3i)} \qquad z = 3i \;\longrightarrow\; \frac{6i+1}{3i(5i)} = C = \frac{1+6i}{-15}$$

$z = 0 \;\Rightarrow\; \frac{1}{(2i)(-3i)} = A = \frac{+1}{6};\; z = -2i \;\longrightarrow\; \frac{-4i+1}{(-2i)(-5i)} = B = \frac{-1+4i}{10}$. Therefore

$$\frac{2z+1}{z^3-iz^2+6z} = \frac{1}{6} \quad \frac{1}{z} + \left(\frac{-1+4i}{10}\right) \frac{1}{z+2i} - \left(\frac{1+6i}{15}\right) \frac{1}{z-3i} \,.$$

Note: $\frac{1}{z}$ **and** $\frac{1}{z+2i}$ **are analytical inside** $C \;\longrightarrow\; \oint_C \frac{dz}{z} = \oint_C \frac{1}{z+2i}\, dz = 0$ **by C.I.T. For**

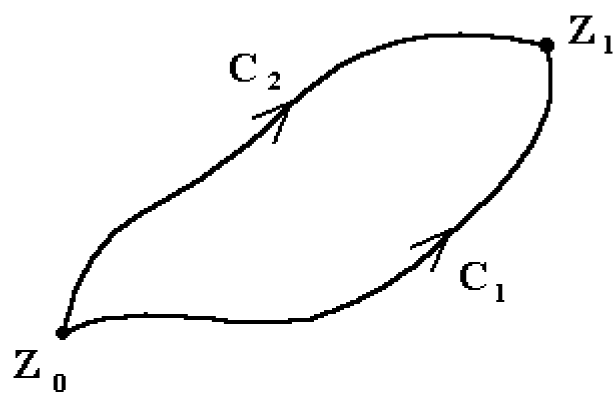
$\oint_C \frac{1}{z-3i} dz.$ **Now** C: $z = 3i + \frac{1}{3}e^{it} \quad 0 \le t \le 2\pi \;\longrightarrow\; \oint_C \frac{dz}{z-3i} = \int_0^{2\pi} \frac{1}{\frac{1}{3}e^{it}} \; \frac{1}{3} i e^{it}$

$dt = \int_0^{2\pi} i\,dt = 2\pi i \;\longrightarrow\; \oint_C \frac{(2z+1)dz}{z^3-iz^2+6z} = -\left(\frac{1+6i}{15}\right) 2\pi i = \frac{\pi}{15}\,(12-2i)$

<u>Some consequences of the Cauchy Integral Theorem</u>

Theorem 1: If $f(z)$ **is analytic in a simply connected domain** D**, then** $\int_C f(z)dz$ **is independent of path in** D**.**

Proof: Let z_0 **and** z_1 **to 2 points in** D **and let** C_1 **and** C_2 **be ny 2 paths from** z_o **to** z_1**. Let** $-C_2$ **denote** C_2 **in other direction and** $C = C_1 - C_2$**. Then** C **is a closed path in** D**.**



By the Cauchy Integral Theorem:

$$\oint_C f(z)dz = 0 \;\Rightarrow\; \int_{C_1} f(z)dz + \int_{-C_2} f(z)dz = 0 \;\Rightarrow$$

$$\int_{C_1} f(z)dz = -\int_{-C_2} f(z)dz = -\int_{C_2} f(z)dz$$

More consequences will be given in Lecture 10.