Ma 530

First Order Differential Equations

Basic Concepts

Classification of Differential Equations

A differential equation is an equation involving an unknown function and one or more of its derivatives. Thus it is a relation of the form

$$F\ (x,y,\frac{dy}{dx},\ldots,\frac{d^n y}{dx^n}) = 0$$

F is given, and we are to find y. The above is an ordinary differential equation of order n.

Example

$$\frac{dy}{dx} = f(x) \quad \text{or} \quad y' = f(x)$$

.

Definition. The order of a differential equation is the order of the highest derivative appearing in the equation. If the equation is a polynomial in the unknown function and its derivatives, then the degree of such an equation is the power to which the highest derivative is raised.

Example $a(x)y'' + b(x)y' + c(x)y = f(x)$. second order, first degree

Remark: $f(x) = 0 \Rightarrow$ homogeneous equation. $f(x) \neq 0$ nonhomogeneous.

Example $(\frac{d^3s}{dt^3})^2 + 5s^4\ t^3 = 0$ 3rd order, 2nd degree

Example $2xy'' - (x+3)y' + 6x^4y = 0$ 2nd order, first degree

Up until now we have mentioned only ordinary differential equations. We shall eventually be concerned with partial differential equations also.

Example

$$uu_{xx} = \frac{1}{c^2}u_t$$

Here $u = u(x,t)$. This is a partial differential equation. Here c is a constant.

Solutions of Differential Equations.

Consider the n-th order ordinary differential equation

$$F(x,y,\ \frac{dy}{dx}\ ,\ldots,\ \frac{d^n y}{dx^n}) = 0. \qquad (1)$$

Definition. A solution of the ordinary differential equation (1) is a real valued function $y(x)$ defined on some interval I such that:

1. $y(x)$ and its first n derivatives exist for each $x \in I$.

2. The substitution of $y(x)$ into the differential equation makes the equation an identity in the interval I.

Remarks: I may be $(-\infty,\infty)$, $[a,b]$, (a,b), $(a,b]$, $[a,b)$. We assume I is not degenerate. Now $1 \Rightarrow y$ and its $n-1$ first derivatives are continuous.

Example $y' = x \quad -\infty < x < \infty \quad y = \frac{1}{2}x^2$ is a solution since

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}x^2\right) = x$$

.

Thus $y(x) = \frac{1}{2}x^2$ is solution for $-\infty < x < \infty$.

The function

$$y = \begin{cases} \frac{1}{2}x^2 - 1 & x \geq 0 \\ \frac{1}{2}x^2 & x < 0 \end{cases}$$

is not a solution of the differential equation on $-\infty < x < \infty$ due to the discontinuity at $x = 0$. $y = \frac{1}{2}x^2 - 1$ is a solution on $0 < x < \infty$ whereas $y = \frac{1}{2}x^2$ is a solution on $-\infty < x < 0$.

Example $y' + y = 0$ One solution is $y = e^{-x}$. The general solution is $y = ce^{-x}$, where c is any constant.

Remarks about solutions:

1. Sometimes we obtain the solution to a differential equation implicitly in the form $f(x,y) = 0$. We need not always solve for y as a function of x (cannot). However, we can verify that we have a solution by implicit differentiation.

Example $e^y \frac{dy}{dx} + x = 0$

$\Rightarrow e^y\, dy + x\, dx = 0$

$\Rightarrow e^y + \frac{x^2}{2} = c$ (*)

We could write $y = \ln(c - \frac{x^2}{2})$ but need not. To see if (*) is a solution we differentiate implicitly. (*) $\Rightarrow e^y \frac{dy}{dx} + x = 0$.

2. Not all equations have solutions.

Example $(y')^2 + y^2 = -1$ has no solution.

Clearly $y = 0$ is not a solution. If $y \neq 0$, $\Rightarrow y^2 > 0$ and $(y')^2 > 0$.

Initial and Boundary Value Problems

We have seen above that a differential equation need not have a unique solution.

Example $y' = x$ $y = \frac{1}{2}x^2 + c$.

If we are given some subsidiary condition then we will "pick" out a unique solution. For example, if we are given the initial conditions $y(0) = -1 \Rightarrow c = -1 \Rightarrow$ and $y = \frac{1}{2}x^2 - 1$.

For first order equations one is given one condition. For second order equations one needs two conditions.

Example $y'' + y = 0$

One may verify directly that $y = c_1 \sin x + c_2 \cos x$ is the solution, where c_1 and c_2 are constants.

If, for example, we are given $y(0) = 0$ and $y'(0) = 1 \Rightarrow$

$y(0) = c_1 \sin 0 + c_2 \cos 0 = c_2 = 0 \Rightarrow y = c_1 \sin x \Rightarrow y'(x) = c_1 \cos x \Rightarrow y'(0) = c_1 = 1$

Thus $y = \sin x$ is the solution.

We could have been given the boundary conditions $y(0) = 0$ $y = (\frac{\pi}{2}) = 2$

$\Rightarrow c_2 = 0$ as before. Also $y(\frac{\pi}{2}) = c_1 \sin\frac{\pi}{2} = 2$ $\Rightarrow c_1 = 2. \Rightarrow y = 2\sin x$

The above are two different kinds of conditions. When the two conditions are given at the *same* point, they are called Initial Conditions; when the two conditions are given at two *different* points, they are called Boundary Conditions.

The equation together with the two conditions is called either an Initial Value Problem (I.V.P.) or a Boundary Value Problem (B.V. P.).

Example

$$\text{DE } y'' = 2x$$

$$\text{B.C. } y(0) = 0 \quad y(2) = 1$$

This is a B.V.P.

$$y' = x^2 + c_1$$

so

$$y = \frac{x^3}{3} + c_1 x + c_2$$

$y(0) = 0 \Rightarrow c_2 = 0$ $\quad$ $y(2) = 1 \Rightarrow \frac{8}{3} + 2c_1 = 1 \Rightarrow 2c_1 = 1 - \frac{8}{3} = -\frac{5}{3}$ and therefore $c_1 = -\frac{5}{6}$

$\Rightarrow$

$$y(x) = \frac{x^3}{3} - \frac{5}{6}x$$

is the solution.

Example

$$\text{D.E.} y'' = 2x$$

$$\text{I.C.} y(1) = 0 \quad y'(1) = -1$$

This is an I.V.P.

$$y' = x^2 + c_1$$

$$\text{so } y(x) = \frac{x^3}{3} + c_1 x + c_2$$

$$y(1) = 0 \quad \Rightarrow \quad \frac{1}{3} + c_1 + c_2 = 0$$

$$y'(1) = -1 \quad \Rightarrow \quad 1 + c_1 = -1$$

$$\Rightarrow c_1 = -2 \quad \text{and} \quad \frac{1}{3} - 2 + c_2 = 0 \quad \Rightarrow c_2 = \frac{5}{3}$$

Thus

$$y(x) = \frac{x^3}{3} - 2x + \frac{5}{3}$$

is the solution.

The Differential Equation $y' = f(x)$

The simplest first order differential equation is $y' = f(x)$. It has the solution

$$y(x) = \int_a^x f(x)dt + c$$

where a and x are in the interval in question.

This allows us to deal with the I.V.P.

$$\text{D.E. } y' = f(x)$$

$$\text{I.C. } y(x_0) = y_0$$

Now $y(x) = \int_{x_0}^{x} f(t)dt + c$ so $y(x_0) = c = y_0 \Rightarrow$

$$y(x) = \int_{x_0}^{x} f(t)dt + y_0$$

.

Formal short way to get above:

$$y' = \frac{dy}{dx} = f(x)$$

We rewrite the equation as

$$dy = f(x)dx$$

integrate from $(x_0, y_0) \to (x, y)$

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(t)dt$$

$$y - y_0 = \int_{x_0}^{x} f(t)dt.$$

Example D.E. $y' = |x| \quad -\infty < x < \infty$ I.C. $y(-1) = 2$

Note that the differential equation may be written as

$$y' = -x \quad -\infty < x < 0 \qquad y' = x \quad 0 \le x < \infty$$

$\Rightarrow$

$$\int_2^y dy = \begin{cases} \int_{-1}^{x} -t\,dt & x < 0 \\ \int_{-1}^{0} -t\,dt + \int_0^x +t\,dt & x > 0 \end{cases}$$

$$y - 2 = \begin{cases} -\frac{t^2}{2}\Big|_{-1}^{x} \\ -\frac{t^2}{2}\Big|_{-1}^{0} + \frac{t^2}{2}\Big|_{0}^{x} \end{cases}$$

$$y - 2 = \begin{cases} -\frac{x^2}{2} + \frac{1}{2} & x < 0 \\ \frac{1}{2} + \frac{x^2}{2} & x \ge 0 \end{cases}$$

Thus

$$y = \begin{cases} -\frac{x^2}{2} + \frac{5}{2} & x < 0 \\ \frac{x^2}{2} + \frac{5}{2} & x \ge 0 \end{cases}$$

Note y is continuous at $x = 0$.