Ma 530

BOUNDARY VALUE PROBLEMS

Homogeneous Boundary Value Problems

Consider the following problem:

$$\left.\begin{aligned} &\textbf{D.E.}\, L[y] = a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \qquad a \le x \le b \\ &\text{B.C. } \alpha_1 y(a) + \beta_1 y'(a) = 0 \qquad \alpha_1^2 + \beta_1^2 \neq 0 \\ &\text{B.C. } \alpha_2 y(b) + \beta_2 y'(b) = 0 \qquad \alpha_2^2 + \beta_2^2 \neq 0 \end{aligned}\right\} \qquad (1)$$

Here $\alpha_1, \alpha_2, \beta_1,$ and β_2 are constants.

Example

$$y'' = 0 \qquad y'(0) = y'(1) = 0$$

$\left(\text{Here } \alpha_1 = \alpha_2 = 0\right)$

$\Rightarrow y = Ax + b \qquad y'(x) = A \qquad y'(0) = y'(1) = A = 0$

$\Rightarrow y(x) = b \qquad b$ any constant.

The Boundary Value Problem (1) is called linear and homogeneous since if $u_1(x)$ and $u_2(x)$ satisfy it, $\Rightarrow c_1 u_1(x) + c_2 u_2(x)$ also does.

Remark. The homogeneous Boundary Value Problem (B.V.P.) always possesses the solution $y(x) = 0$.

Question. When does there exit a nonzero solution to (1)?

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of $L[y] = 0$. $\Rightarrow y(x) = c_1 y_1 + c_2 y_2$ is the general solution of the DE.

$$\text{B.C. } \Rightarrow \left.\begin{aligned} \alpha_1 y(a) + \beta_1 y'(a) = 0 \\ \alpha_2 y(b) + \beta_2 y'(b) = 0 \end{aligned}\right\} \qquad \text{and } y(x) = c_1 y_1 + c_2 + y_2 \Rightarrow$$

$$c_1[\alpha_1 y_1(a) + \beta_1 y_1'(a)] + c_2[\alpha_1 y_2(a) + \beta_1 y_2'(a)] = 0$$
$$c_1[\alpha_2 y_1(b) + \beta_2 y_1'(b)] + c_2[\alpha_2 y_2(b) + \beta_2 y_2'(b)] = 0.$$

The above are two equations for c_1 and c_2. We want a nontrivial solution. Let $B_a(u) = \alpha_1 u(a) + \beta_1 u'(a)$ and $B_b(u) = \alpha_2 u(b) + \beta_2 u'(b)$. Then the determinant of the coefficients of the above system must equal zero. Thus we require

$$\begin{vmatrix} B_a(y_1) & B_a(y_2) \\ B_b(y_1) & B_b(y_2) \end{vmatrix} = 0 \qquad (2)$$

Theorem 1. The homogeneous linear B.V.P. (1) has a nontrivial solution if and if (2) holds.

Theorem 2. If $u(x)$ is a particular nontrivial solution of the B.V.P. (1), then all solutions are given by $y = cu(x)$ where c is an arbitrary constant.

Proof. Let $v(x)$ be any solution, $u(x)$ a particular solution of the B.V.P. (1) $\Rightarrow \alpha_1 u(a) + \beta_1 u'(a) = 0$ and
$\alpha_1 v(a) + \beta_1 v'(a) = 0$ since u and v both satisfy the first B.C. These equations may be regarded as equations for α_1, β_1. However, since by assumption α_1 and β_1 are not both zero $\Rightarrow$

$$\begin{vmatrix} u(a) & u'(a) \\ v(a) & v'(a) \end{vmatrix} = 0 = W[u,v]_{x=a} \Rightarrow W[u(x), v(x)] = 0 \text{ for } a \le x \le b$$

$\Rightarrow$ u and v are two LD solutions of the D.E. $\Rightarrow$ there exist constants $c_1, c_2 \neq 0$ such that $c_1 u(x) + c_2 v(x) = 0$ for $a \le x \le b \Rightarrow v(x) = -\frac{c_1}{c_2} u(x) = cu(x)$.

Example

$$y'' - \lambda^2 y = 0 \qquad \lambda \neq 0 \qquad y(0) = y(1) = 0$$

The general solution is $y = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$. The B.C. $y(0) = 0. \Rightarrow c_1 + c_2 = 0,$ whereas the condition $y(1) = 0$ leads to
$c_1 e^{\lambda} + c_2 e^{-\lambda} = 0.$ The two equations for c_1 and c_2 are

$$c_1 + c_2 = 0$$
$$c_1 e^{\lambda} + c_2 e^{-\lambda} = 0$$

The determinant of the coefficients is $\begin{vmatrix} 1 & 1 \\ e^{\lambda} & e^{-\lambda} \end{vmatrix} 1 \neq 0. \Rightarrow c_1 = c_2 = 0 \Rightarrow$ the only solution is $y \equiv 0$.

Example

$$y'' + \lambda y = 0 \qquad \lambda > 0 \qquad y(\pi) = y(2\pi) = 0$$

The general solution of the D.E. is $y = c_1 \sin\sqrt{\lambda}\, x + c_2 \cos\sqrt{\lambda}\, x$

B.Cs. $\Rightarrow$

$$0 = c_1 \sin\sqrt{\lambda}\,\pi + c_2 \cos\sqrt{\lambda}\,\pi$$
$$0 = c_1 \sin 2\sqrt{\lambda}\,\pi + c_2 \cos 2\sqrt{\lambda}\,\pi$$

The determinant of the system = $\begin{vmatrix} \sin\sqrt{\lambda}\,\pi & \cos\sqrt{\lambda}\,\pi \\ \sin 2\sqrt{\lambda}\,\pi & \cos 2\sqrt{\lambda}\,\pi \end{vmatrix} = \sin\sqrt{\lambda}\,\pi \cos 2\sqrt{\lambda}\,\pi - \cos\sqrt{\lambda}\,\pi \sin\sqrt{\lambda}\,\pi$

$$= +\sin(\sqrt{\lambda}\,\pi - 2\sqrt{\lambda}\,\pi) = -\sin\sqrt{\lambda}\,\pi = 0$$

This holds $\Leftrightarrow$ $\sqrt{\lambda} = n$, where $n = 1,2\ldots$ $\quad \Rightarrow \lambda = n^2 = 1,4\ldots$
Therefore we get a non-zero solution only for these values of λ.

Consider the system of equations for $\lambda = 1$

$$\Rightarrow 0 = c_1 \cdot 0 + c_2 \cdot -1$$
$$0 = c_1 \cdot 0 + c_2 \cdot 1 \Rightarrow c_2 = 0,\ c_1 \text{ arbitrary}$$

Thus $y = c_1 \sin x$ is the solution for $\lambda = 1$, where c_1 any constant. This is an example of our next topic.

Eigenvalue Problems

The following special kind of B.V.P. is called an eigenvalue problem.

$$\left.\begin{array}{ll} L[y] + \lambda y = 0 & a \le x \le b \\ \text{B.C. } \alpha_1 y(a) + \beta_1 y'(a) = 0 & \alpha_1^2 + \beta_1^2 \neq 0 \\ \text{B.C. } \alpha_2 y(b) + \beta_2 y'(b) = 0 & \alpha_2^2 + \beta_2^2 \neq 0 \end{array}\right\} (*)$$

Here $L[y] = a_0(x)y'' + a_1(x)y' + a_2(x)y$, and λ is a parameter.

Again $y \equiv 0$ is a solution for all λ. However, we are interested in nontrivial (nonzero) solutions.

Definition. If a nontrivial solution of the B.V.P. $(*)$ exists for a value $\lambda = \lambda_i$, then λ_i is called an eigenvalue of L (relevant to the B.Cs.). The corresponding nontrivial solution $y_i(x)$ is called an eigenfunction.

Remark. $y'' + \lambda y = 0 \qquad y(\pi) = y(2\pi) = 0$. Here $L[y] = y''$

This example was discussed above for the case $\lambda > 0$ The eigenvalues are $\lambda = 1,4,\ldots$ and the eigenfunctions are $\sin x, \sin 2x,\ldots$
Note $\lambda = 0$ is not an eigenvalue since $\lambda = 0 \Rightarrow y'' = 0 \Rightarrow y = Ax + B$
$y(\pi) = 0 \Rightarrow \pi A + B = 0 \qquad y(2\pi) = 0 \Rightarrow 2\pi A + B = 0 \Rightarrow A = B = 0$. Also, the case $\lambda < 0$ leads to $y = 0$.
Note that $c_1 \sin \lambda x$, where λ is one of the eigenvalues, is also an eigenfunction (*for all*) $c_1 \neq 0$.

Remark. $\lambda = 1 \Rightarrow \sin x \qquad \lambda = 4 \Rightarrow \sin 2x$
In addition,

$$\int_{\pi}^{2\pi} \sin x \sin 2x dx = 2\int_{\pi}^{2\pi} \sin^2 x \cos x dx = \frac{2}{3}\sin^3 x|_{\pi}^{2\pi} = 0.$$

This is a special case of the following general result.

Theorem. Let λ_n and λ_m be distinct eigenvalues and y_n and y_m corresponding eigenfunctions for

$$y'' + \lambda y = 0 \qquad y(0) = y(L) = 0.$$

Then

$$\int_0^L y_n(x) y_m(x) dx = 0 \qquad n \neq m.$$

Proof.

$$y_n'' + \lambda_n y_n = 0 \qquad \times y_m$$
$$y_m'' + \lambda_m y_m = 0 \qquad \times y_n$$

Subtracting and then integrating the result we get
$\Rightarrow$

$$\int_0^L (\lambda_n - \lambda_m) y_n y_m dx + \int_0^L (y_n'' y_m - y_m'' y_n) dx = 0$$

$\Rightarrow$

$$(\lambda_n - \lambda_m) \int_0^L y_n(x) y_m(x) dx = \int_0^L (y_m'' y_n - y_n'' y_m) dx$$

We now integrate the first integral on the right by parts twice and use the boundary conditions.

$$\int_0^L (y_m'' y_n) dx = y_n(x) y_m'(x)|_0^L - \int_0^L y_m' y_n' dx = -y_n'(x) y_m(x)|_0^L + \int_0^L y_n'' y_m dx$$
$$= \int_0^L y_n'' y_m dx$$

since $y(0) = y(L) = 0$.
Therefore

$$(\lambda_n - \lambda_m) \int_0^L y_m(x) y_n(x) dx = 0.$$

Example Find the eigenvalues and eigenfunctions for

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \qquad y(1) = 0$$

We must consider three cases; $\lambda < 0, \lambda = 0,$, and $\lambda > 0$.

I. $\lambda < 0$. Let $\lambda = -\alpha^2$ where $\alpha \neq 0$. Then the differential equation becomes

$$y'' - \alpha^2 y = 0$$

and has the general solution

$$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}.$$

The boundary conditions $\Rightarrow$
$y'(0) = c_1\alpha - c_2\alpha = 0$ or $c_1 = c_2$, and $y(1) = c_1 e^{\alpha} + c_2 e^{-\alpha} = 0$.$\Rightarrow$ $c_1 = c_2 = 0$.
Thus for $\lambda < 0$, the only solution is $y = 0$.

II. $\lambda = 0$. The solution is $y = c_1 x + c_2$. The BCs imply $c_1 = c_2 = 0$. Again the only solution is $y = 0$.
III. $\lambda > 0$. Let $\lambda = \beta^2$ where $\beta \neq 0$. The DE becomes

$$y'' + \beta^2 y = 0$$

and has the general solution

$$y = c_1 \sin \beta x + c_2 \cos \beta x.$$

The BCs imply

$$y'(0) = c_1 \beta \cos 0 - c_2 \beta \sin 0 = c_1 \beta = 0$$

Hence $c_1 = 0,$, since $\beta \neq 0.$ Thus

$$y = c_2 \cos \beta x.$$

Now $y(1) = c_2 \cos \beta = 0$. Since we want a nontrivial solution we cannot have $c_2 = 0$. Hence

$$\cos \beta = 0 \Rightarrow \beta = \frac{2n+1}{2}\pi, n = 0, \pm 1, \pm 2, \ldots$$

We therefore have the eigenvalues

$$\lambda_n = \left(\frac{2n+1}{2}\right)^2 \pi^2,$$

and eigenfunctions

$$y_n(x) = C_n \cos\left(\frac{2n+1}{2}\right)x,$$

for $n = 0, 1, 2, \ldots$ Note the negative values of n do not give additional eigenfunctions since $\cos(-t) = \cos t$.