Homogeneous Boundary Value Problems

Consider the following problem:

\[
\begin{cases}
\text{D.E.} & L[y] = a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \quad a \leq x \leq b \\
\text{B.C.} & a_1y(a) + \beta_1y'(a) = 0 \quad a_1^2 + \beta_1^2 \neq 0 \\
& a_2y(b) + \beta_2y'(b) = 0 \quad a_2^2 + \beta_2^2 \neq 0
\end{cases}
\]

(1)

Here \(a_1, a_2, \beta_1, \text{ and } \beta_2\) are constants.

Example

\[
y'' = 0 \quad y'(0) = y'(1) = 0
\]

(Here \(a_1 = a_2 = 0\))

\[
\Rightarrow y = Ax + b \quad y'(x) = A \quad y'(0) = y'(1) = A = 0
\]

\[
\Rightarrow y(x) = b \quad \text{any constant.}
\]

The Boundary Value Problem (1) is called linear and homogeneous since if \(u_1(x)\) and \(u_2(x)\) satisfy it, \(c_1u_1(x) + c_2u_2(x)\) also does.

Remark. The homogeneous Boundary Value Problem (B.V.P.) always possesses the solution \(y(x) = 0\).

Question. When does there exit a nonzero solution to (1)?

Let \(y_1(x)\) and \(y_2(x)\) be two linearly independent solutions of \(L[y] = 0\). \Rightarrow y(x) = c_1y_1 + c_2y_2\) is the general solution of the DE.

B.C. \Rightarrow

\[
\begin{cases}
& a_1y_1(a) + \beta_1y_1'(a) = 0 \\
& a_2y_2(b) + \beta_2y_2'(b) = 0
\end{cases}
\]

and \(y(x) = c_1y_1 + c_2 + y_2 \Rightarrow \)

\[
\begin{align*}
c_1[a_1y_1(a) + \beta_1y_1'(a)] + c_2[a_1y_2(a) + \beta_1y_2'(a)] &= 0 \\
c_1[a_2y_1(b) + \beta_2y_1'(b)] + c_2[a_2y_2(b) + \beta_2y_2'(b)] &= 0.
\end{align*}
\]

The above are two equations for \(c_1\) and \(c_2\). We want a nontrivial solution. Let \(B_a(u) = a_1u(a) + \beta_1u'(a)\) and \(B_b(u) = a_2u(b) + \beta_2u'(b)\). Then the determinant of the coefficients of the above system must equal zero. Thus we require
Theorem 1. The homogeneous linear B.V.P. (1) has a nontrivial solution if and if (2) holds.

Theorem 2. If \( u(x) \) is a particular nontrivial solution of the B.V.P. (1), then all solutions are given by \( y = cu(x) \) where \( c \) is an arbitrary constant.

Proof. Let \( v(x) \) be any solution, \( u(x) \) a particular solution of the B.V.P. (1) \( \Rightarrow \alpha_1u(a) + \beta_1u'(a) = 0 \) and
\[
\alpha_1v(a) + \beta_1v'(a) = 0
\]
since \( u \) and \( v \) both satisfy the first B.C. These equations may be regarded as equations for \( \alpha_1, \beta_1. \) However, since by assumption \( \alpha_1 \) and \( \beta_1 \) are not both zero \( \Rightarrow \)
\[
\begin{vmatrix}
  u(a) & u'(a) \\
  v(a) & v'(a)
\end{vmatrix} = 0 = W[u, v]_{x=a} \Rightarrow W[u(x), v(x)] = 0 \text{ for } a \leq x \leq b
\]
\( \Rightarrow u \) and \( v \) are two LD solutions of the D.E. \( \Rightarrow \) there exist constants \( c_1, c_2 \neq 0 \) such that \( c_1u(x) + c_2v(x) = 0 \) for \( a \leq x \leq b \) \( \Rightarrow v(x) = -\frac{c_1}{c_2}u(x) = cu(x). \)

**Example**

\[
y'' - \lambda^2y = 0 \quad \lambda \neq 0 \quad y(0) = y(1) = 0
\]
The general solution is \( y = c_1e^{\lambda x} + c_2e^{-\lambda x}. \) The B.C. \( y(0) = 0, \Rightarrow c_1 + c_2 = 0, \) whereas the condition \( y(1) = 0 \) leads to \( c_1e^\lambda + c_2e^{-\lambda} = 0. \) The two equations for \( c_1 \) and \( c_2 \) are
\[
\begin{align*}
c_1 + c_2 &= 0 \\
c_1e^\lambda + c_2e^{-\lambda} &= 0
\end{align*}
\]
The determinant of the coefficients is
\[
\begin{vmatrix}
  1 & 1 \\
  e^\lambda & e^{-\lambda}
\end{vmatrix} \neq 0. \Rightarrow c_1 = c_2 = 0 \Rightarrow \text{the only solution is } y = 0.
\]

**Example**

\[
y'' + \lambda y = 0 \quad \lambda > 0 \quad y(\pi) = y(2\pi) = 0
\]
The general solution of the D.E. is \( y = c_1 \sin \sqrt{\lambda}x + c_2 \cos \sqrt{\lambda}x \)
B.Cs. \( \Rightarrow \)
\[
\begin{align*}
0 &= c_1 \sin \sqrt{\lambda} \pi + c_2 \cos \sqrt{\lambda} \pi \\
0 &= c_1 \sin 2\sqrt{\lambda} \pi + c_2 \cos 2\sqrt{\lambda} \pi
\end{align*}
\]
The determinant of the system
\[
\begin{vmatrix}
  \sin \sqrt{\lambda} \pi & \cos \sqrt{\lambda} \pi \\
  \sin 2\sqrt{\lambda} \pi & \cos 2\sqrt{\lambda} \pi
\end{vmatrix} = \sin \sqrt{\lambda} \pi \cos 2\sqrt{\lambda} \pi - \cos \sqrt{\lambda} \pi \sin 2\sqrt{\lambda} \pi
\]
\[ = + \sin(\sqrt{\lambda} \pi - 2\sqrt{\lambda} \pi) = -\sin \sqrt{\lambda} \pi = 0 \]

This holds \( \sqrt{\lambda} = n \), where \( n = 1, 2, \ldots \) \( \Rightarrow \lambda = n^2, 1, 4, \ldots \)

Therefore we get a non-zero solution only for these values of \( \lambda \).

Consider the system of equations for \( \lambda = 1 \)
\[
\Rightarrow 0 = c_1 \cdot 0 + c_2 \cdot -1
\]
\[
0 = c_1 \cdot 0 + c_2 \cdot 1 \Rightarrow c_2 = 0, \text{ } c_1 \text{ arbitrary}
\]

Thus \( y = c_1 \sin x \) is the solution for \( \lambda = 1 \), where \( c_1 \) any constant. This is an example of our next topic.

**Eigenvalue Problems**

The following special kind of B.V.P. is called an eigenvalue problem.

\[
L[y] + \lambda y = 0 \\
\text{B.C. } a_{1y}(a) + \beta_{1y}(a) = 0 \\
\text{B.C. } a_{2y}(b) + \beta_{2y}(b) = 0
\]

Here \( L[y] = a_0(x)y'' + a_1(x)y' + a_2(x)y \), and \( \lambda \) is a parameter.

Again \( y = 0 \) is a solution for all \( \lambda \). However, we are interested in nontrivial (nonzero) solutions.

**Definition.** If a nontrivial solution of the B.V.P. \( (*) \) exists for a value \( \lambda = \lambda_i \), then \( \lambda_i \) is called an eigenvalue of \( L \) (relevant to the B.Cs.). The corresponding nontrivial solution \( y_i(x) \) is called an eigenfunction.

**Remark.** \( y'' + \lambda y = 0 \) \( y(\pi) = y(2\pi) = 0 \). Here \( L[y] = y'' \)

This example was discussed above for the case \( \lambda > 0 \) The eigenvalues are \( \lambda = 1, 4, \ldots \) and the eigenfunctions are \( \sin x, \sin 2x, \ldots \)

Note \( \lambda = 0 \) is not an eigenvalue since \( \lambda = 0 \Rightarrow y'' = 0 \Rightarrow y = Ax + B \)

\( y(\pi) = 0 \Rightarrow \pi A + B = 0 \quad y(2\pi) = 0 \Rightarrow 2\pi A + B = 0 \Rightarrow A = B = 0 \). Also, the case \( \lambda < 0 \) leads to \( y = 0 \).

Note that \( c_1 \sin \lambda x \), where \( \lambda \) is one of the eigenvalues, is also an eigenfunction (for all) \( c_1 \neq 0 \).

**Remark.** \( \lambda = 1 \Rightarrow \sin x \quad \lambda = 4 \Rightarrow \sin 2x \)

In addition,
\[
\int_{\pi}^{2\pi} \sin x \sin 2xdx = 2 \int_{\pi}^{2\pi} \sin^2 x \cos xdx = \frac{2}{3} \sin x|_{\pi}^{2\pi} = 0.
\]

This is a special case of the following general result.
Theorem. Let $\lambda_n$ and $\lambda_m$ be distinct eigenvalues and $y_n$ and $y_m$ corresponding eigenfunctions for

$$y'' + \lambda y = 0 \quad y(0) = y(L) = 0.$$ 

Then

$$\int_0^L y_n(x)y_m(x)dx = 0 \quad n \neq m.$$ 

Proof.

$$y'' + \lambda_n y_n = 0 \times y_m$$

$$y'' + \lambda_m y_m = 0 \times y_n$$

Subtracting and then integrating the result we get

$$\Rightarrow \quad \int_0^L (\lambda_n - \lambda_m) y_n y_m dx + \int_0^L (y''_n y_m - y''_m y_n) dx = 0$$

$$\Rightarrow \quad (\lambda_n - \lambda_m) \int_0^L y_n(x)y_m(x) dx = \int_0^L (y''_m y_n - y''_n y_m) dx$$

We now integrate the first integral on the right by parts twice and use the boundary conditions.

$$\int_0^L (y''_m y_n) dx = y_n(x)y'_m(x)|_0^L - \int_0^L y'_m y'_n dx = -y'_n(x)y_m(x)|_0^L + \int_0^L y''_m y_m dx$$

$$= \int_0^L y''_m y_m dx$$

since $y(0) = y(L) = 0$.

Therefore

$$\Rightarrow \quad (\lambda_n - \lambda_m) \int_0^L y_m(x)y_n(x) dx = 0.$$ 

**Example** Find the eigenvalues and eigenfunctions for

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) = 0$$

We must consider three cases; $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.

I. $\lambda < 0$. Let $\lambda = -\alpha^2$ where $\alpha \neq 0$. Then the differential equation becomes

$$y'' - \alpha^2 y = 0$$

and has the general solution

$$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}.$$ 

The boundary conditions $\Rightarrow$

$$y'(0) = c_1 \alpha - c_2 \alpha = 0 \quad \text{or} \quad c_1 = c_2, \quad \text{and} \quad y(1) = c_1 e^\alpha + c_2 e^{-\alpha} = 0 \quad \Rightarrow \quad c_1 = c_2 = 0.$$ 

Thus for $\lambda < 0$, the only solution is $y = 0$.

II. $\lambda = 0$. The solution is $y = c_1 x + c_2$. The BCs imply $c_1 = c_2 = 0$. Again the only solution is $y = 0$.

III. $\lambda > 0$. Let $\lambda = \beta^2$ where $\beta \neq 0$. The DE becomes
and has the general solution
\[ y = c_1 \sin \beta x + c_2 \cos \beta x. \]
The BCs imply
\[ y'(0) = c_1 \beta \cos 0 - c_2 \beta \sin 0 = c_1 \beta = 0 \]
Hence \( c_1 = 0, \) since \( \beta \neq 0. \) Thus
\[ y = c_2 \cos \beta x. \]
Now \( y(1) = c_2 \cos \beta = 0. \) Since we want a nontrivial solution we cannot have \( c_2 = 0. \)
Hence
\[ \cos \beta = 0 \Rightarrow \beta = \frac{2n + 1}{2} \pi, n = 0, \pm 1, \pm 2, \ldots \]
We therefore have the eigenvalues
\[ \lambda_n = \left( \frac{2n + 1}{2} \right)^2 \pi^2, \]
and eigenfunctions
\[ y_n(x) = C_n \cos\left( \frac{2n + 1}{2} \right) x, \]
for \( n = 0, 1, 2, \ldots \) Note the negative values of \( n \) do not give additional eigenfunctions since \( \cos(-t) = \cos t. \)