Ma 530

Calculus of Variations II

Calculus of Variations with Constraints

We begin with some examples.

Example 1

What curve through the points (x_1, y_1) and $(x_2 . y_2)$ of given length L has the maximum area between the curve and the x –axis?

If $y(x)$ is a **single-valued function** of x, then

$$A[y(x)] = \int_{x_1}^{x_2} y(x)dx \tag{1}$$

whereas

$$L = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx \tag{2}$$

Therefore the problem is to maximize (1) subject to the constraint (2) and the conditions $y(x_1) = y_1, y(x_2) = y_2$.

Example 2

If in the previous example we do not assume that $y(x)$ is a single-valued function of x, then it is convenient to suppose that x and y are given parametrically, i.e.,

$$x = x(t), \quad y = y(t) \quad t_1 \leq t \leq t_2$$

where $x(t_1) = x_1, x(t_2) = x_2, y(t_1) = y_1$, and $y(t_2) = y_2$. Then we have the constraint

$$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = L \tag{3}$$

Also

$$dA = \frac{1}{2}(xdy - ydx)$$

since by Green's Theorem

$$\oint_C Pdx + Qdy = \iint_R (Q_x - P_y)dA$$

so that

$$\oint_C \frac{1}{2}(xdy - ydx) = \frac{1}{2}\iint_R [1 - (-1)]dA = \iint_R dA = \text{area of } R$$

Thus

$$A[x(t),y(t)] = \frac{1}{2}\int_{t_1}^{t_2}\left[x\frac{dy}{dt} - y\frac{dx}{dt}\right]dt \tag{4}$$

The problem is to minimize (4) subject to the constraint (3) and the conditions $x(t_1) = x_1, x(t_2) = x_2$, $y(t_1) = y_1$, and $y(t_2) = y_2$.

The above two examples illustrate problems in which one desires to minimize (or maximize) a given integral subject to a constraint. Several examples of such problems are

$$\delta\int_{x_1}^{x_2} F(x,y,y')dx = 0 \text{ subject to } \int_{x_1}^{x_2} G(x,y,y')dx = \text{constant} \tag{5}$$

$$\delta\int_{x_1}^{x_2} F\left(t,x,y,\dot{x},\dot{y}\right)dx = 0 \text{ subject to } \int_{x_1}^{x_2} G\left(t,x,y,\dot{x},\dot{y}\right)dx = \text{constant} \tag{6}$$

$$\delta\int_{x_1}^{x_2} F(x,u,v,u',v')dx = 0 \text{ subject to } \phi(u,v) = 0 \tag{7}$$

where $u = u(x), v = v(x)$.

We deal with (5) first. Thus our problem is to make

$$I[y(x)] = \int_{x_1}^{x_2} F(x,y,y')dx = \text{minimum or maximum}$$

where y is prescribed as $y(x_1) = y_1$ and $y(x_2) = y_2$ subject to the condition

$$J[y(x)] = \int_{x_1}^{x_2} G(x,y,y')dx = k$$

where k is a given constant.

This problem cannot be attacked by the earlier method of forming $y + \epsilon\eta$ where η vanishes on the boundary only, for in general such functions do not satisfy the subsidiary condition in a neighborhood of $\epsilon = 0$ except at $\epsilon = 0$. Since we have **two** requirements, we therefore consider the function

$$y(x) + \epsilon_1\eta_1(x) + \epsilon_2\eta_2(x)$$

where η_1 and η_2 have continuous derivatives and

$$\eta_1(x_1) = \eta_1(x_2) = \eta_2(x_1) = \eta_2(x_2) = 0.$$

Then we have

$$\Phi(\epsilon_1,\epsilon_2) = I[y(x) + \epsilon_1\eta_1(x) + \epsilon_2\eta_2(x)] = \int_{x_1}^{x_2} F(x, y + \epsilon_1\eta_1 + \epsilon_2\eta_2, y' + \epsilon_1\eta_1' + \epsilon_2\eta_2')dx$$

subject to

$$\Psi(\epsilon_1,\epsilon_2) = J[y(x) + \epsilon_1\eta_1(x) + \epsilon_2\eta_2(x)] - k = \int_{x_1}^{x_2} G(x, y + \epsilon_1\eta_1 + \epsilon_2\eta_2, y' + \epsilon_1\eta_1' + \epsilon_2\eta_2')dx - k = 0$$

Thus we desire that the function $\Phi(\epsilon_1,\epsilon_2)$ take on a relative minimum or maximum at all $\epsilon_1 = \epsilon_2 = 0$ with respect to all sufficiently small values of ϵ_1 and ϵ_2 for which $\Psi(\epsilon_1,\epsilon_2) = 0$.

This problem is of the form we treated earlier by means of Lagrange multipliers. In particular, if λ is the Lagrange multiplier, then we require

$$\left.\frac{\partial(\Phi + \lambda\Psi)}{\partial\epsilon_1}\right|_{\epsilon_1=\epsilon_2=0} = 0 \tag{1}$$

$$\left.\frac{\partial(\Phi + \lambda\Psi)}{\partial\epsilon_2}\right|_{\epsilon_1=\epsilon_2=0} = 0 \tag{2}$$

and

$$\Psi(\epsilon_1,\epsilon_2) = 0$$

Now (1) $\Rightarrow$

$$\int_{x_1}^{x_2}[F_y\eta_1 + F_{y'}\eta_1']dx + \lambda\int_{x_1}^{x_2}[G_y\eta_1 + G_{y'}\eta_1']dx = 0$$

and (2) $\Rightarrow$

$$\int_{x_1}^{x_2}[F_y\eta_2 + F_{y'}\eta_2']dx + \lambda\int_{x_1}^{x_2}[G_y\eta_2 + G_{y'}\eta_2']dx = 0$$

Let

$$[H]_y = H_y - \frac{d}{dx}H_{y'}$$

Then as before an integration by parts yields

$$\int_{x_1}^{x_2}\left\{[F]_y + \lambda[G]_y\right\}\eta_1 dx = 0 \tag{3}$$

and

$$\int_{x_1}^{x_2}\left\{[F]_y + \lambda[G]_y\right\}\eta_2 dx = 0 \tag{4}$$

If $[G]_y \neq 0$, then we can, say, choose η_2 such that $\int_{x_1}^{x_2}[G]_y\eta_2 dx \neq 0$, and thus λ may be chosen so that (4) holds. However, since η_1 is arbitrary, λ will not be such that (3) holds. Therefore it follows from (3) that

$$[F]_y + \lambda[G]_y = 0$$

or

$$\frac{\partial(F+\lambda G)}{\partial y} - \frac{d}{dx}\left(\frac{\partial(F+\lambda G)}{\partial y'}\right) = 0$$

is the necessary condition. The general solution of this equation will involve two constants of integration and the constant parameter λ. Thus we have 3 constants to satisfy the 3 conditions $y(x_1) = y_1, y(x_2) = y_2$, and $\int_{x_1}^{x_2}G(x,y,y')dx = k$.

The above results may be summarized as follows:

Theorem. In order to minimize (or maximize) $\int_{x_1}^{x_2}Fdx$ subject to a constraint $\int_{x_1}^{x_2}Gdx = k$ we first write $H = F + \lambda G$, where λ is a constant, and minimize (or maximize) $\int_{x_1}^{x_2}Hdx$ subject to no constraints. Carry the Lagrange multiplier λ through the calculations, and determine it, together with the constants of integration arising in the Euler equation, so that the constraint $\int_{x_1}^{x_2}Gdx = k$ holds, and the end conditions are satisfied.

Example Maximize

$$A[y(x)] = \int_{x_1}^{x_2}y(x)dx \tag{1}$$

subject to

$$L = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx \tag{2}$$

and $y(x_1) = y_1, y(x_2) = y_2$.

Solution: Without loss of generality we may move the axis and change the scale so that the curve is to pass through $(0, 0)$ and $(1, 0)$. Thus we must maximize

$$\int_0^1 y(x)dx$$

subject to $y(0) = y(1) = 0$ and the constraint

$$\int_0^1 \sqrt{1 + (y')^2}\, dx = L \quad \text{where } L > 1$$

We form

$$H = y + \lambda\left(1 + (y')^2\right)^{\frac{1}{2}}$$

The Euler equation

$$-H_y + \frac{d}{dx} H_{y'} = 0$$

implies

$$-1 + \lambda \frac{d}{dx}\left\{ \frac{y'}{\left(1 + (y')^2\right)^{\frac{1}{2}}} \right\} = 0$$

Then integrating we have

$$\lambda \frac{y'}{\left(1 + (y')^2\right)^{\frac{1}{2}}} = x - c_1$$

or

$$\left\{\lambda^2 - (x - c_1)^2\right\}(y')^2 = (x - c_1)^2$$

Therefore

$$y' = \pm \frac{(x - c_1)}{\sqrt{\lambda^2 - (x - c_1)^2}}$$

so

$$y = \pm\left(\lambda^2 - (x - c_1)^2\right)^{\frac{1}{2}} + c_2$$

or finally

$$(y - c_2)^2 + (x - c_1)^2 = \lambda^2$$

The required curves are arcs of circles. We have three constants to determine. They are determined so that the arc passes through $(0, 0), (0, 1)$ and has length L.

Remark: When $L = \frac{\pi}{2}$, we have a semicircle since the circumference of the circle is $2\pi\lambda$. For $L > \pi/2$ y is no longer a single-valued function of x. For such a case it is convenient to employ a parametric representation expressing x and y as functions of t, i.e., $x = x(t)$ and $y = y(t)$. We are led to the problem

Maximize

$$I = \frac{1}{2}\int_{t_1}^{t_2}\left(x\,\dot{y} - y\,\dot{x}\right)dt$$

where $x(t_1) = 0, x(t_2) = 1, y(t_1) = y(t_2) = 0$ and

$$J = \int_{t_1}^{t_2}\left[\left(\dot{x}\right)^2 + \left(\dot{y}\right)^2\right]^{\frac{1}{2}}dt = L$$

Remark: It is easily seen that the problem

$$\delta\int_{x_1}^{x_2} F\left(t,x,y,\dot{x},\dot{y}\right)dx = 0 \text{ subject to } \int_{x_1}^{x_2} G\left(t,x,y,\dot{x},\dot{y}\right)dx = \text{constant}$$

leads to the Euler equations

$$\frac{d}{dt}H_{\dot{x}} - H_x = 0$$
$$\frac{d}{dt}H_{\dot{y}} - H_y = 0$$

where $H = F + \lambda G$.
For our problem

$$H = \frac{1}{2}\left(x\,\dot{y} - y\,\dot{x}\right) + \lambda\left(\dot{x}^2 + \dot{y}^2\right)^{\frac{1}{2}}$$

These again leads to arcs of circles.

Remark: For the problem

$$\delta\int_{x_1}^{x_2} F(x,u,v,u',v')dx = 0 \text{ subject to } \phi(u,v) = 0$$

the Euler equations are

$$\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) - \frac{\partial F}{\partial u} - \lambda\phi = 0$$
$$\frac{d}{dx}\left(\frac{\partial F}{\partial v'}\right) - \frac{\partial F}{\partial v} - \lambda\phi = 0$$