Ma 530

Calculus of Variations I

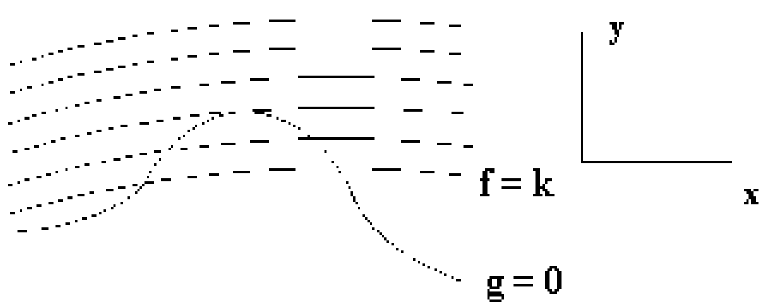
We begin by recalling some results from maxima-minima problems in calculus.

The Method of Lagrange Multipliers

Suppose we want to find the stationary values of a function $f(x,y)$, that is, the points at which a function $f(x,y)$ might have either a maximum or minimum, in the case when the two variables x and y are not mutually independent, but are connected by a constraint of the form $g(x,y) = 0$.

Example Suppose one wants to find the rectangle with given area 16 that has the smallest perimeter. Then if x and y are the dimensions of the rectangle, we want to maximize $f(x,y) = 2(x+y)$ subject to the fact that $xy = 16$. That is we want to maximize $f = 2(x+y)$ subject to the condition $g(x,y) = xy - 16 = 0$.

Suppose that the $g(x,y) = 0$ curve and the level curves $f(x,y) = k$ curves in the x,y –plane look as below.



As we describe the curve $g = 0$ we encounter curves $f = k$, and in general k changes monotonically, i.e., either increases or decreases. At the point where the sense in which we run through the k –scale is reversed we may expect an extremum value. From the figure this will occur at a point (x_0, y_0) where the $f = k$ curve and the $g = 0$ curve have tangents that are parallel. Since the normal is orthogonal to the tangent vector, this means that the normals to these curves will have the same direction. Thus at (x_0, y_0)

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

where λ is a constant of proportionality. Thus we have the two conditions

$$f_x - \lambda g_x = 0$$
$$f_y - \lambda g_y = 0$$

at the extremum point (x_0, y_0). We also have the third condition $g(x,y) = 0$. These three conditions allow us to solve for the three unknowns $x_0, y_0,$ and λ. The constant λ is called a *Lagrange multiplier.*

We may state Lagrange's Rule:
To find the extreme values of the function $f(x,y)$ subject to the subsidiary condition $g(x,y) = 0$,
(a) Find all values of x, y, λ such that

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

and

$$g(x,y) = 0$$

(b) Evaluate f at all the points (x,y) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Example (Same as above.) Suppose one wants to find the rectangle with given area 16 that has the largest perimeter.
If x and y are the dimensions of the rectangle, we want to maximize $f(x,y) = 2(x+y)$ subject to the fact that $xy = 16$. That is we want to maximize $f = 2(x+y)$ subject to the condition $g(x,y) = xy - 16 = 0$. The condition $\nabla f = \lambda \nabla g$ leads to

$$\langle 2, 2 \rangle = \langle \lambda y, \lambda x \rangle$$

Thus we have that $x = y$. Then $g = 0$ implies that $x = y = 4$, which is a square.

The method of Lagrange Multipliers may be applied to functions of three or more variables.
Example Find the points on

$$4x - 5y + 3z = 2$$

which are closest to $(1, -2, 3)$

Solution:
The distance is given by:

$$\sqrt{(x-1)^2 + (y+2)^2 + (z-3)^2}$$

It is easier to work with the square of the distance. Thus we want to minimize

$$f(x,y,z) = (x-1)^2 + (y+2)^2 + (z-3)^2$$

subject to the constraint

$$g(x,y,z) = 4x - 5y + 3z - 2 = 0$$

Let

$$\begin{aligned}\nabla f &= \langle 2(x-1), 2(y+2), 2(z-3)\rangle \\ &= \lambda \nabla g \\ &= \langle 4\lambda, -5\lambda, 3\lambda\rangle\end{aligned}$$

Then

$$\begin{aligned}\frac{x-1}{2} &= \lambda \\ \frac{2(y+2)}{-5} &= \lambda \\ \frac{2(z-3)}{3} &= \lambda\end{aligned}$$

Thus

$$\frac{x-1}{2} = \frac{2(z-3)}{3}$$

Solution is: $\left\{x = -3 + \frac{4}{3}z\right\}$
and

$$\frac{2(y+2)}{-5} = \frac{2(z-3)}{3}$$

, Solution is: $\left\{y = 3 - \frac{5}{3}z\right\}$

But the point (x,y,z) lies on the plane and must satisfy $g(x,y,z) = 4x - 5y + 3z - 2 = 0,$ so

$$4\left(-3 + \frac{4}{3}z\right) - 5\left(3 - \frac{5}{3}z\right) + 3z = 2$$

Solution is: $\left\{z = \frac{87}{50}\right\}$ Thus

$$x = -3 + \frac{4}{3}\left(\frac{87}{50}\right) = -\frac{17}{25}$$

$$y = 3 - \frac{5}{3}\left(\frac{87}{50}\right) = \frac{1}{10}$$

Example Find the extreme values of $f(x,y) = xy$ on the circle $x^2 + y^2 = 1$. Let $g(x.y) = x^2 + y^2 - 1$. We have the equation

$$\langle y,x\rangle = \langle 2\lambda x, 2\lambda y\rangle$$

which leads to the three equations

$$\begin{aligned}y - 2\lambda x &= 0 \\ x - 2\lambda y &= 0 \\ x^2 + y^2 &= 1\end{aligned}$$

, Solution is: $\left\{\lambda = \frac{1}{2}, y = \rho_1, x = \rho_1\right\}, \left\{\lambda = -\frac{1}{2}, y = \rho_1, x = -\rho_1\right\}$ where ρ_1 is a root of $2\hat{Z}^2 - 1$,
Thus $x_0 = y_0 = \pm\frac{\sqrt{2}}{2}$. We have the following results

x_0	y_0	$f(x_0,y_0)$	
$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	maximum
$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	minimum
$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	minimum
$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	maximum

Example Suppose one wants to cut a beam with maximal rectangular cross section from a circular log of radius $\sqrt{2}$.

Solution:

We shall use Lagrange multipliers to show that the optimal beam has square cross section. Let the origin be at the center of the log and the beam so that the x –axis cuts the log and beam in half horizontally and the y –axis cuts the log and beam in half vertically. (This means that the beam has dimensions $2x$ by $2y$.) The log satisfies the equation

$$x^2 + y^2 = 2$$

If (x,y) is the corner of the beam in first quadrant, then we must maximize the area

$$A = f(x,y) = 4xy$$

of the beam's rectangular cross section subject to the constraint

$$g(x,y) = x^2 + y^2 - 2 = 0$$

Then

$$\nabla f = \langle 4y, 4x\rangle = \lambda\nabla g = \langle 2\lambda x, 2\lambda y\rangle$$

so that

$$\lambda = \frac{2y}{x} = \frac{2x}{y}$$

or

$$x^2 = y^2$$

But the fact that the corner of the beam must lie on the log, that is the circle $x^2 + y^2 = 2,$ tells us that

$$x^2 + x^2 = 2$$

so $x = y = 1.$ (Recall that (x,y) is in the first quadrant.) Thus the beam is square with dimensions $2x = 2y = 2.$

The Method of Lagrange Multipliers with Two Constraints

For optimization problems involving two constraint functions g and $h,$ one must introduce a second Lagrange multiplier, $\mu,$ and solve the equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

as illustrated in the next example.

Example Optimization with two constraints

Let

$$T(x,y,z) = 20 + 2x + 2y + z^2$$

represent the temperature at each point on the sphere

$$x^2 + y^2 + z^2 = 11$$

Find the extreme temperatures on the curve formed by the intersection of the plane

$$x + y + z = 3$$

and the sphere.

Solution: Here we have two constraints

$$g(x,y,z) = x^2 + y^2 + z^2 - 11$$

and

$$h(x,y,z) = x + y + z - 3$$

We therefore need two constants λ and μ.

$$\begin{aligned} \nabla T &= \langle 2, 2, 2z \rangle \\ \lambda \nabla g &= \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle \\ \mu \nabla h &= \langle \mu, \mu, \mu \rangle \end{aligned}$$

Thus

$$\langle 2, 2, 2z \rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle + \langle \mu, \mu, \mu \rangle$$

We then have the system of five equations in five unknowns

$$\begin{aligned} 2 &= 2\lambda x + \mu \\ 2 &= 2\lambda y + \mu \\ 2z &= 2\lambda z + \mu \\ x^2 + y^2 + z^2 &= 11 \\ x + y + z &= 3 \end{aligned}$$

Subtracting the second equation from the first yields the system

$$\begin{aligned} \lambda(x - y) &= 0 \\ 2z(1 - \lambda) - \mu &= 0 \\ x^2 + y^2 + z^2 &= 11 \\ x + y + z &= 3 \end{aligned}$$

The first equation tells that $\lambda = 0$ or $x = y$. Consider each case separately.

If $\lambda = 0$, then since $2 = 2\lambda x + \mu$ we have that $\mu = 2$. Using this in the second equation we see that $z = 1$. the last two equations become

$$x^2 + y^2 = 10 \quad \text{and} \quad x + y = 2$$

Substitution yields

$$x^2 + (2 - x)^2 = 10$$

or

$$x^2 - 2x - 3 = 0$$

which has the solutions $x = 3, x = -1$. The corresponding y values are $y = -1, y = 3$, respectively. Thus if $\lambda = 0$ the critical points are $(3, -1, 1)$ and $(-1, 3, 1)$.

If $\lambda \neq 0$, then $x = y$ and the last two equations become

$$2x^2 + z^2 = 11$$
$$2x + z = 3$$

By substitution

$$2x^2 + (3 - 2x)^2 = 3$$

or

$$3x^2 - 6x - 1 = 0$$

, Solution is: $\left\{x = 1 + \frac{2}{3}\sqrt{3}\right\}, \left\{x = 1 - \frac{2}{3}\sqrt{3}\right\}$. The corresponding values of z are $z = \left(3 \mp 4\sqrt{3}/3\right)$, so we have two more critical points.

To find the optimal solutions, one must compute the temperatures at the four critical points.

$$T(3, -1, 1) = T(-1, 3, 1) = 25$$
$$T\left(\frac{3 - 2\sqrt{3}}{3}, \frac{3 - 2\sqrt{3}}{3}, \frac{3 + 4\sqrt{3}}{3}\right) \approx 30.333$$
$$T\left(\frac{3 + 2\sqrt{3}}{3}, \frac{3 + 2\sqrt{3}}{3}, \frac{3 - 4\sqrt{3}}{3}\right) \approx 30.333$$

Thus $T = 25$ is the minimum temperature and $T = 33.333$ is the maximum temperature on the curve.

Introductory Remarks and Examples

The basic problem in the calculus of variations is to determine a function such that a certain definite integral involving that function and certain of its derivatives takes on a maximum or minimum value.

Example 1

Consider the family of curves $y(x)$ through the points $P_1(x_1, y_1)$, and $P_2(x_2, y_2)$ with continuous derivatives. The length of such a curve is given by

$$L = \int_{P_1}^{P_2} ds = \int_{P_1}^{P_2} \sqrt{(dx)^2 + (dy)^2}$$

Thus

$$L(y(x)) = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

where $y(x)$ is such that $y(x_1) = y_1$ and $y(x_2) = y_2$.
Question: What function $y(x)$ makes L a minimum?

Remark: Note that we have conditions on $y(x)$.

Example 2

Determine the curve through (x_1, y_1) and (x_2, y_2) such that when it is rotated about the x –axis it gives the minimum surface area.

$$\Delta A = 2\pi y ds$$

so that

$$A(y(x)) = 2\pi \int_{x_1}^{x_2} y ds$$

$$A(y(x)) = 2\pi \int_{P_1}^{P_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

where $y(x)$ is such that $y(x_1) = y_1$ and $y(x_2) = y_2$ and $y'(x)$ is continuous.

Example 3 The Brachistochrone Problem of Bernoulli

Two points $P_1(x,y)$ and $P_2(x_2,y_2)$ $(y_1 > y_2 > 0)$ are to be connected by a curve along which a frictionless mass point moves in the **shortest possible time** from P_1 to P_2 under gravity acting in the y –direction.
The initial velocity of the mass point is zero. Then

$$T = \int_{P_1}^{P_2} dt = \int_{P_1}^{P_2} \frac{ds}{v} = \int_{x_1}^{x_2} \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{v} dx$$

where v is the velocity of the particle and s is arc length.
After falling a distance $y_1 - y$, we have

$$\frac{1}{2} m v^2 = mg(y_1 - y)$$

$$v^2 = 2g(y_1 - y)$$

$$T(y(x)) = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \left(\frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{y_1 - y}} \right) dx$$

Thus we want to find the curve $y(x)$ that minimizes T and is such that $y(x_1) = y_1 \quad y(x_2) = y_2$ and $y'(x)$ is continuous.

The Simplest Case

Consider now the problem of determining a twice continuously differentiable function $y(x)$ for which the integral

$$I[y(x)] = \int_{x_1}^{x_2} F\big(x, y(x), y'(x)\big) dx \tag{1}$$

takes on a maximum or a minimum value, where $y(x)$ satisfies the prescribed end conditions

$$y(x_1) = y_1 \quad y(x_2) = y_2$$

The function F is twice continuously differentiable with respect to its three arguments x, y, y'

Definition. The **admissible class** of functions for (1) consists of all functions $y(x)$ that are continuous and have continuous first and second derivatives and also satisfy $y(x_1) = y_1, y(x_2) = y_2$.

Problem: Among all admissible $y(x)$ find a necessary condition that $I[y(x)]$ be a maximum or minimum.

Suppose that $y(x)$ is the actual minimizing function and consider $\eta(x)$ (with continuous first and second derivatives) with the properties that

$$\eta(x_1) = \eta(x_2) = 0$$

Then for any constant ϵ, the function

$$y(x) + \epsilon\eta(x)$$

is admissible.

If ϵ is sufficiently small, then all functions $y(x) + \epsilon\eta(x)$ will lie in a small neighborhood of $y(x)$.

Now

$$I[y(x) + \epsilon\eta(x)] = \int_{x_1}^{x_2} F\big(x, y(x) + \epsilon\eta, y'(x) + \epsilon\eta'\big) dx$$
$$= \Phi(\epsilon)$$

since y and η are fixed. Clearly

$$\Phi(\epsilon) \geq \Phi(0)$$

since by assumption $y(x)$ is the minimizing function. A necessary condition for a minimum at $\epsilon = 0$ is

$$\left.\frac{d\Phi}{d\epsilon}\right|_{\epsilon=0} = 0$$

Thus

$$\lim_{\epsilon\to 0}\frac{\Phi(\epsilon)-\Phi(0)}{\epsilon} = 0$$

Now

$$\left.\frac{d\Phi}{d\epsilon}\right|_{\epsilon=0} = \lim_{\epsilon\to 0}\frac{\Phi(\epsilon)-\Phi(0)}{\epsilon} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{x_1}^{x_2}\left\{F\left(x,y(x)+\epsilon\eta,y'(x)+\epsilon\eta'\right)-F\left(x,y,y'\right)\right\}dx$$

The generalized mean value theorem says

$$F(x,s+\Delta s,t+\Delta t)-F(x,s,t) = \frac{\partial F}{\partial s}(x,s+\theta_1\Delta s,t+\theta_2\Delta t)\Delta s+\frac{\partial F}{\partial t}(x,s+\theta_1\Delta s,t+\theta_2\Delta t)\Delta t$$

where $0 < \theta_1,\theta_2 < 1$. Therefore

$$\begin{aligned}\left.\frac{d\Phi}{d\epsilon}\right|_{\epsilon=0} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{x_1}^{x_2}\left\{F_y\left(x,y+\epsilon\theta_1\eta,y'+\epsilon\theta_2\eta'\right)\epsilon\eta+F_{y'}\left(x,y+\epsilon\theta_1\eta,y'+\epsilon\theta_2\eta'\right)\epsilon\eta'\right\}dx\\ &= \int_{x_1}^{x_2}\left[F_y\left(x,y,y'\right)\eta+F_{y'}\left(x,y,y'\right)\eta'\right]dx\end{aligned}$$

Integrating the last term by parts $\Rightarrow$

$$\left.\frac{d\Phi}{d\epsilon}\right|_{\epsilon=0} = \int_{x_1}^{x_2}\left[F_y-\frac{d}{dx}F_{y'}\right]\eta dx+\left[F_{y'}\eta\right]_{x_1}^{x_2} = 0$$

Since $\eta(x_1) = \eta(x_2) = 0,$ the last term above vanishes and we have

$$\int_{x_1}^{x_2}\left[F_y-\frac{d}{dx}F_{y'}\right]\eta dx = 0$$

Since η is arbitrary, by the Fundamental Lemma of the Calculus of Variations (see below), we have derived the necessary condition

$$F_y-\frac{d}{dx}F_{y'} = 0 \qquad (*)$$

Equation $(*)$ is called Euler's equation. It is a necessary condition that a function $y(x)$ must satisfy if it minimizes $\Phi(\epsilon)$ at $\epsilon = 0$.

Fundamental Lemma of the Calculus of Variations.

If the relation

$$\int_{x_1}^{x_2}\phi(x)\eta(x)dx = 0,$$

where $\phi(x)$ is continuous, holds for all functions $\eta(x)$ which vanish on the boundary and have continuous second derivatives, then $\phi(x) \equiv 0$ in $[x_1,x_2]$.

Proof. Suppose $\phi(x) \neq 0$. Then at some point $x = c$, $\phi(x) > 0$, say. Thus $\exists$ a $\delta > 0$, such that for $x \in (c-\delta,c+\delta)$, $\phi(x) > 0$. Let $\eta(x) = [x-(c-\delta)]^4[x-(c+\delta)]^4$ in $(c-\delta,c+\delta)$, and zero otherwise. Then

$$\int_{x_1}^{x_2}\phi(x)\eta(x)dx = \int_{c-\delta}^{c+\delta}\phi(x)\eta(x)dx > 0$$

since we are integrating a positive function over a positive range. This is a contradiction and therefore $\phi(x) \equiv 0$ in $[x_1,x_2]$.

Exercise: Apply Euler's equation to Example 1 above and show that a straight line satisfies the necessary condition.

Remarks: (1) In general $F = F(x,y,y')$ and the Euler eq. $F_y - \frac{d}{dx}F_{y'} = 0$ written out is

$$F_y - \frac{\partial}{\partial x}F_{y'} - \frac{\partial}{\partial y}F_{y'}\frac{dy}{dx} - \frac{\partial}{\partial y'}F_{y'}\frac{dy'}{dx} = 0$$

or

$$F_y - F_{y'x} - F_{y'y}\frac{dy}{dx} - F_{y'y'}\frac{d^2y}{dx^2} = 0$$

This is a second order ordinary D.E. for y. Solving this subject to the boundary conditions $y(x_1) = y(x_2) = 0$ yields the solution.

(2) Special Cases of Interest

A.

$$F = F(x,y')$$

The Euler Equation is then

$$\frac{d}{dx}F_{y'} = 0$$

and thus becomes

$$\frac{\partial F}{\partial y'} = \text{constant}$$

B.

$$F = F(y,y')$$

Here the Euler Eq. is

$$F_y - F_{y'y}\frac{dy}{dx} - F_{y'y'}\frac{d^2y}{dx^2} = 0$$

It may be shown that

$$F_y - F_{y'y}\frac{dy}{dx} - F_{y'y'}\frac{d^2y}{dx^2} = \frac{1}{y'}\frac{d}{dx}\left\{F - y'F_{y'}\right\}$$

Therefore for this case the Euler equation can be integrated to get

$$F - y'F_{y'} = c, \; c \text{ a constant}$$

Remark: Example 2 above, namely, minimize

$$A(y(x)) = 2\pi\int_{P_1}^{P_2} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \;\text{ subject to } y(x_1) = y_1, y(x_2) = y_2$$

may be solved using the integrated Euler equation given in Case B.

Example 3 The Brachistochrone Problem of Bernoulli

The problem sated above is essentially to minimize

$$T(y(x)) = \int_{x_1}^{x_2} \left(\frac{\sqrt{1 + \left(y'\right)^2}}{\sqrt{y_1 - y}} \right) dx \text{ subject to } y(x_1) = y_1, y(x_2) = y_2$$

Here

$$F = F\left(y, y'\right) = \frac{\sqrt{1 + \left(y'\right)^2}}{\sqrt{y_1 - y}}$$

We may use the result from Case B above.

$$F_{y'} = \frac{1}{2} \frac{2y'}{\sqrt{1 + \left(y'\right)^2} \sqrt{y_1 - y}}$$

Thus the equation $y' F_{y'} - F = c$ leads to

$$\frac{\left(y'\right)^2}{\sqrt{1 + \left(y'\right)^2} \sqrt{y_1 - y}} - \frac{\sqrt{1 + \left(y'\right)^2}}{\sqrt{y_1 - y}} = c$$

or

$$\left(y'\right)^2 - \left(1 + \left(y'\right)^2\right) = c\sqrt{1 + \left(y'\right)^2} \sqrt{y_1 - y}$$

This may be written as

$$\frac{1}{y_1 - y} - c^2 = c^2 \left(y'\right)^2$$

so

$$\frac{\frac{1}{c^2} - (y_1 - y)}{y_1 - y} = \left(y'\right)^2$$

or

$$\frac{dy}{dx} = \pm \sqrt{\frac{a - (y_1 - y)}{y_1 - y}} \quad \text{where } a = \frac{1}{c^2}$$

This first order DE is separable

$$dx = \pm \sqrt{\frac{y_1 - y}{a - (y_1 - y)}}\, dy$$

To integrate this let $y_1 - y = a \sin^2 \frac{t}{2}$. Then

$$dx = \pm \frac{\sqrt{a}\, \sin \frac{t}{2}}{\sqrt{a}\, \cos \frac{t}{2}} \left(-a \sin \frac{t}{2} \cos \frac{t}{2}\right) dt$$

$$dx = \pm a \sin^2 \frac{t}{2}\, dt$$

Therefore

$$\int_{x_1}^{x} dx = \int \pm a \sin^2 \frac{t}{2}\, dt = \pm a \int_{x_1}^{x} \frac{1 - \cos t}{2}\, dt$$

so that

$$x - x_1 = \pm\frac{a}{2}(t - \sin t)$$
$$y_1 - y = a\sin^2\frac{t}{2} = a\left(\frac{1-\cos t}{2}\right)$$

These last two equations are parametric equations for x and y as a function of the parameter t. We choose a such that when $x = x_2$ $y = y_2$.
The above curve is a piece of a cycloid with cusp at (x_1, y_1).

Natural Boundary Conditions

Recall that we showed that a necessary condition for

$$\int_{x_1}^{x_2} F\left(x, y, y'\right) dx$$

to be a minimum was that

$$\frac{d\Phi}{d\epsilon}\Big|_{\epsilon=0} = \int_{x_1}^{x_2}\left[F_y - \frac{d}{dx}F_{y'}\right]\eta dx + \left[F_{y'}\eta\right]_{x_1}^{x_2} = 0 \tag{1}$$

For the case that we considered the unknown function $y(x)$ was specified at x_1 and x_2, and therefore it was necessary to choose $\eta(x)$ such that $\eta(x_1) = \eta(x_2) = 0$ in order to have $y + \epsilon\eta$ satisfy $y(x_1) = y_1$ and $y(x_2) = y_2$. When $y(x)$ is **not** preassigned at the endpoints, then $\eta(x)$ need not vanish there. However, (1) must still hold. Furthermore, equation (1) must hold for all permissible η's, and therefore the coefficient of η in the integral must be zero. Hence we are led to the requirement that

$$\left[F_{y'}\eta\right]_{x_1}^{x_2} = 0$$

when y is not prescribed at x_1 and x_2. Since $\eta(x_1)$ and $\eta(x_2)$ are now arbitrary, this implies that

$$F_{y'}\Big|_{x=x_1} = 0 \text{ and/or } F_{y'}\Big|_{x=x_2} = 0$$

These are called *Natural Boundary Conditions.*

If y is prescribed only at $x = x_1$, then the second condition holds, but not the first. Similarly, if y is prescribed at $x = x_2$ then the first condition only holds.

The Variational Notation

Consider a function $G(x, y, z)$. If we let

$$\Delta G = G(x + \Delta x, y + \Delta y, z + \Delta z) - G(x, y, z)$$

then the differential

$$dG = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y + \frac{\partial G}{\partial z}\Delta z$$

represents an approximation to ΔG if $\Delta x, \Delta y,$ and Δz are small.

In the calculus of variations we considered an integrand of the form $F\left(x, y, y'\right)$. For a **fixed value of** x this depends on the **function** $y(x)$ **and its derivatives.** In our derivation of the Euler equation for F we changed $y(x)$ into the new function $y + \epsilon\eta$. This is similar to the change $x + \Delta x, y + \Delta y, z + \Delta z$ above. In particular we are seeking an analogy to the differential dG. Note, however, that while dG represents change in G **along a particular curve,** we shall be concerned with changes in F from **curve to curve** (corresponding to changes in y for fixed x).

The change in $y(x)$, namely $\epsilon\eta(x)$, is called the variation of y and is denoted by

$$\delta y \equiv \epsilon\eta$$

Corresponding to this change in $y(x)$ for a fixed value of x, F changes by ΔF where

$$\Delta F = F\left(x, y(x) + \epsilon\eta, y'(x) + \epsilon\eta'\right) - F\left(x, y, y'\right)$$

By Taylor's expansion

$$\Delta F = \left\{F_y(\epsilon\eta) + F_{y'}\left(\epsilon\eta'\right)\right\} + \frac{1}{2!}\left\{F_{yy}(\epsilon\eta)^2 + F_{y'y'}\left(\epsilon\eta'\right)^2 + 2F_{yy'}(\epsilon\eta)\left(\epsilon\eta'\right)\right\} + \cdots$$

In analogy with the definition of the differential, we define

$$\delta F = F_y(\epsilon\eta) + F_{y'}\left(\epsilon\eta'\right)$$

as the **first variation** of F. Note that letting $F = y' \Rightarrow \delta y' = \epsilon\eta'$ so that

$$\delta F = F_y \delta y + F_{y'} \delta y'$$

Remark: For a complete analogy one might expect that

$$\delta F = F_x \delta x + F_y \delta y + F_{y'} \delta y'$$

However, since x remains fixed $\delta x = 0$.

Some Properties of δ :
Given $F_1\left(x, y, y'\right)$ and $F_2\left(x, y, y'\right)$, then

1. $\delta(F_1 + F_2) = \delta F_1 + \delta F_2$

2. $\delta(F_1 F_2) = F_1 \delta F_2 + F_2 \delta F_1$

3. $\delta\left\{\frac{F_1}{F_2}\right\} = \frac{F_2 \delta F_1 - F_1 \delta F_2}{(F_2)^2}$

Lemma. If x is the independent variable (and accordingly $\delta x = 0$) then the operators δ and $\frac{d}{dx}$ commute, i.e.

$$\frac{d}{dx}(\delta y) = \delta \frac{dy}{dx}$$

Proof: $\delta y = \epsilon\eta \Rightarrow$

$$\frac{d}{dx}(\delta y) = \frac{d}{dx}(\epsilon\eta) = \epsilon\eta' = \delta y' = \delta \frac{dy}{dx}$$

Remark: The above lemma is not generally true unless the differentiation is with respect to an independent variable.

Lemma. If

$$I[y] = \int_{x_1}^{x_2} F\left(x, y, y'\right) dx$$

Then

$$\delta I = \delta \int_{x_1}^{x_2} F\left(x,y,y'\right)dx = \int_{x_1}^{x_2} \delta F dx$$

Proof.

$$\begin{aligned}
\Delta I &= \int_{x_1}^{x_2} \left\{F\left(x,y(x)+\epsilon\eta,y'(x)+\epsilon\eta'\right) - F\left(x,y,y'\right)\right\}dx \\
&= \int_{x_1}^{x_2} \Delta F dx = \int_{x_1}^{x_2} \left(\delta F + \delta^2 F + \cdots\right)dx \\
&= \int_{x_1}^{x_2} \delta F dx + \int_{x_1}^{x_2} \delta^2 F dx + \cdots
\end{aligned}$$

Thus

$$\delta I = \int_{x_1}^{x_2} \delta F dx = \int_{x_1}^{x_2} \left[F_y(\epsilon\eta) + F_{y'}\left(\epsilon\eta'\right)\right]dx$$

Note: $\Delta F = \delta F + \delta^2 F + \cdots$ and $\Delta I = \delta I + \delta^2 I + \cdots$

Theorem. The integral

$$I[y] = \int_{x_1}^{x_2} F\left(x,y,y'\right)dx$$

is stationary if and only if it first variation vanishes, i.e.

$$\delta I = \delta \int_{x_1}^{x_2} F\left(x,y,y'\right)dx = 0$$

for every possible variation δy.

Remark. From the Theorem it follows that a stationary function for an integral is one for which the variation of that integral is zero, whereas a stationary point of a function is one at which the differential of the function is zero.

Example: Let

$$I[y] = \int_a^b \left[y^2 + \left(y'\right)^2 + xyy'\right]dx \quad y(a) = y_1, y(b) = y_2$$

What are the conditions on $y(x)$ to make I stationary?

Solution:

$$\delta I = \int_{x_1}^{x_2} \left[2y\delta y + 2y'\delta y' + xy\delta y' + xy'\delta y\right]dx = 0$$

where δy vanishes at the end points x_1 and x_2.

$$\delta I = \int_{x_1}^{x_2} \left[2y\delta y + 2y'\frac{d}{dx}\delta y + xy\frac{d}{dx}\delta y + xy'\delta y\right]dx = 0$$

Integrating by parts the second term in the integrand above we have

$$\int_{x_1}^{x_2} 2y'\frac{d}{dx}\delta y dx = 2y'\delta y\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} 2\frac{d}{dx}y'\delta y dx = -\int_{x_1}^{x_2} 2y''\delta y dx$$

Similarly we integrate the third term in the integrand. Combining we get

$$\int_{x_1}^{x_2} \left[2y - 2y'' - (xy)' + xy'\right]\delta y dx = 0$$

Thus the Euler equation for this problem is

$$2y - 2y'' - (xy)' + xy' = 0$$

Some Generalizations

(1) **Higher derivatives of the dependent variable:**
Consider

$$I[y] = \int_{x_1}^{x_2} F\left(x, y, y', y''\right) dx$$

where

$$y(x_1) = y_1 \qquad y(x_2) = y_2$$
$$y'(x_1) = y_1' \qquad y'(x_2) = y_2'$$

What function minmizes or maximizes the above integral and satisfies the boundary condition?

If $y(x)$ is the minimizing function, then we consider $y(x) + \epsilon\eta(x)$ where

$$\eta(x_1) = \eta(x_2) = \eta'(x_1) = \eta'(x_2) = 0$$

Now $\delta I = 0$ will yield a stationary condition.

$$\delta I = \int_{x_1}^{x_2} \left[F_y \delta y + F_{y'} \delta y' + F_{y''} \delta y'' \right] dx = 0$$

$$\int_{x_1}^{x_2} F_{y''} \delta y'' dx = \int_{x_1}^{x_2} F_{y''} \frac{d^2}{dx^2} \delta y dx = F_{y''} \frac{d}{dx} \delta y' \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} F_{y''} \frac{d}{dx} \left(\delta y'\right) dx$$
$$= F_{y''} \frac{d}{dx} \delta y' \Big|_{x_1}^{x_2} - F_{y''} \delta y \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{d^2}{dx^2} F_{y''} \delta y dx = 0$$

Since y and y' are prescribed at x_1 and x_2 then $\delta y = \delta y' = 0$ at x_1 and x_2. Thus

$$\delta I = \int_{x_1}^{x_2} \left[F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} \right] \delta y dx = 0$$

Therefore the Euler equation for this problem is

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} = 0$$

where

$$y(x_1) = y_1 \qquad y(x_2) = y_2$$
$$y'(x_1) = y_1' \qquad y'(x_2) = y_2'$$

In a similar fashion one can show the for the integral

$$I[y] = \int_{x_1}^{x_2} F\left(x, y, y', y'', y''', \ldots, y^{(n)}\right) dx$$

where $y, y', y'', y''', \ldots, y^{(n-1)}$ are specified at x_1 and x_2, that the Euler equation is

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} - \frac{d^3}{dx^3} F_{y'''} + \cdots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0$$

(2)**Several Dependent Variables**

Consider

$$I[x, y_1, y_2, y_3, y_4, \ldots, y_n] = \int_{x_1}^{x_2} F\Big(x, y_1, y_2, \ldots y_n, y_1', y_2', \ldots y_n'\Big)$$

where

$$y_i(x_1) = y_i^{(1)} \qquad y_i(x_2) = y_i^{(1)} \quad i = 1, 2, 3, \ldots, n$$

If $y_i(x)$, for $i = 1, 2, \ldots, n$ are the minimizing functions, we consider $y_i + \epsilon\eta_i$ for $i = 1, 2, \ldots, n$ where $\eta_i(x_1) = \eta_i(x_2) = 0$ for $i = 1, 2, \ldots, n$. We form

$$\begin{aligned}\Phi[\epsilon_1, \ldots, \epsilon_n] &= I[x, y_1 + \epsilon_1\eta_1, y_2 + \epsilon_2\eta_2, y_3 + \epsilon_3\eta_3, \ldots, y_n] \\ &= \int_{x_1}^{x_2} F\Big(x, y_1 + \epsilon_1\eta_1, y_2 + \epsilon_2\eta_2, \ldots y_n + \epsilon_n\eta_n, y_1' + \epsilon_1\eta_1', y_2' + \epsilon_2\eta_2', \ldots y_n' + \epsilon_n\eta_n\Big)dx\end{aligned}$$

Then $\Phi[\epsilon_1, \ldots, \epsilon_n] \geq \Phi[0, \ldots, 0]$ so we have the n conditions

$$\left.\frac{\partial\Phi}{\partial\epsilon_i}\right|_{\epsilon_1=\epsilon_2=\cdots=\epsilon_n=0} = 0 \text{ for } i = 1, 2, \ldots, n$$

Thus

$$\left.\frac{\partial\Phi}{\partial\epsilon_i}\right|_{\epsilon_1=\epsilon_2=\cdots=\epsilon_n=0} = \int_{x_1}^{x_2}\Big[F_{y_i\eta_i} + F_{y_i}'\eta_i\Big]dx = 0 \text{ for } i = 1, 2, \ldots, n$$

We are thus led to n Euler equations

$$F_{y_i} - \frac{d}{dx}F_{y_i'} = 0 \quad \text{for } i = 1, 2, \ldots, n$$