Ma 530

Euler's Equation

The differential equation

$$x^2y'' + pxy' + qy = f(x) \quad (1)$$

where p and q are constants is called Euler's Equation (or the Cauchy-Euler Equation).

Consider the homogeneous case

$$x^2y'' + pxy' + qy = 0. \quad (2)$$

Once we find y_h then we can find y_p for (1) by variation of parameters. We consider (2) only for the case $x > 0$ so that coefficient on y'' does not vanish. Notice that each term contains some power of x if we try $y = x^m$. Hence we seek a homogeneous solution of the form

$$y_h = x^m$$

and shall try to determine m so that x^m is a solution. Now

$$y_h' = mx^{m-1} \text{ and } y_h'' = m(m-1)x^{m-2}$$

so that the differential equation ⇒

$$x^m[m(m-1) + pm + q] = 0 \Rightarrow x^m[m^2 + m(p-1) + q] = 0.$$

Since $x^m \neq 0 \Rightarrow$

$$m^2 + m(p-1) + q = 0. \quad (3)$$

(3) is call the indicial equation for m. It has solutions

$$m = \frac{-(p-1) \pm \sqrt{(p-1)^2 - 4q}}{2}$$

Let $\triangle = (p-1)^2 - 4q$ Then we have three cases just as we had for second order equations with constant coefficients.

Case 1. $\triangle > 0 \Rightarrow$ 2 distinct roots $m_1, m_2 \Rightarrow$

$$y_h = c_1x^{m_1} + c_2x^{m_2}.$$

Case 2. $\triangle = 0 \Rightarrow$

$$m_1 = \frac{1-p}{2}.$$

To get a second solution let $y = u(x)x^{m_1}$, where $u(x)$ is a function to be determined. Equation (2) ⇒

$$xu'' + u' = 0.$$

Letting $v = u' \Rightarrow xv' + v = 0 \Rightarrow$

$$v' + \frac{v}{x} = 0.$$

The integrating factor for this equation is $e^{\int \frac{1}{x}dx} = e^{\ln x} = x \Rightarrow \frac{d}{dx}(xv) = 0 \Rightarrow xv = c_1 \Rightarrow v = \frac{c_1}{x}$

$u = \int v = c_1 \ln x + c_2$

$\Rightarrow$

$$y_h = x^{m_1}[c_1 \ln x + c_2].$$

Case 3. $\triangle < 0 \Rightarrow$ roots are complex conjugates, $m_1 = a + bi$ and $m_2 = a - bi$. Thus

$$y_h = c_1 x^{m_1} + c_2 x^{m_2} = c_1 x^{a+bi} + c_2 x^{a-bi}$$

or

$$y_h = x^a[c_1 x^{bi} + c_2 x^{-bi}].$$

Now

$$x^{a+bi} = e^{(a+ib)\ln x} \text{ for } x > 0.$$

$\Rightarrow$

$$x^{bi} = e^{ib\ln x} = \cos(b\ln x) + i\sin(b\ln x)$$

$\Rightarrow$

$$y_h = x^a[A\cos(b\ln x) + B\sin(b\ln x)].$$

Example Solve

$$x^2 y'' + 7xy' + 5y = 0$$

Here $p = 7$ and $q = 5$. The indicial equation (3) is for this example

$$m^2 + m(p-1) + q = m^2 + 6m + 5 = (m+5)(m+1)$$

$\Rightarrow m = -5$ or $-1 \Rightarrow$

$$y = c_1 x^{-5} + c_2 x^{-1} = \frac{c_1}{x^5} + \frac{c_2}{x}.$$