Ma 530

**Euler’s Equation**

The differential equation

\[ x^2y'' + pxy' + qy = f(x) \]  \hspace{1cm} (1)

where \( p \) and \( q \) are constants is called Euler’s Equation (or the Cauchy-Euler Equation).

Consider the homogeneous case

\[ x^2y'' + pxy' + qy = 0. \]  \hspace{1cm} (2)

Once we find \( y_h \) then we can find \( y_p \) for (1) by variation of parameters. We consider (2) only for the case \( x > 0 \) so that coefficient on \( y'' \) does not vanish. Notice that each term contains some power of \( x \) if we try \( y = x^m \). Hence we seek a homogeneous solution of the form

\[ y_h = x^m \]

and shall try to determine \( m \) so that \( x^m \) is a solution. Now

\[ y_h' = mx^{m-1} \text{ and } y_h'' = m(m-1)x^{m-2} \]

so that the differential equation \( \Rightarrow \)

\[ x^m[m(m-1) + pm + q] = 0 \Rightarrow x^m[m^2 + m(p - 1) + q] = 0. \]

Since \( x^m \neq 0 \) \( \Rightarrow \)

\[ m^2 + m(p - 1) + q = 0. \]  \hspace{1cm} (3)

(3) is called the indicial equation for \( m \). It has solutions

\[ m = \frac{-(p - 1) \pm \sqrt{(p - 1)^2 - 4q}}{2} \]

Let \( \Delta = (p - 1)^2 - 4q \). Then we have three cases just as we had for second order equations with constant coefficients.

**Case 1.** \( \triangle > 0 \) \( \Rightarrow \) 2 distinct roots \( m_1, m_2 \) \( \Rightarrow \)

\[ y_h = c_1x^{m_1} + c_2x^{m_2}. \]

**Case 2.** \( \Delta = 0 \) \( \Rightarrow \)

\[ m_1 = \frac{1 - p}{2}. \]

To get a second solution let \( y = u(x)x^{m_1} \), where \( u(x) \) is a function to be determined. Equation (2) \( \Rightarrow \)

\[ xu'' + u' = 0. \]

Letting \( v = u' \) \( \Rightarrow \)

\[ xv' + v = 0 \Rightarrow \]

\[ v' + \frac{v}{x} = 0. \]

The integrating factor for this equation is

\[ e^{\int \frac{1}{x} dx} = e^{\ln x} = x \Rightarrow \frac{d}{dx}(xy) = 0 \Rightarrow xy = c_1 \Rightarrow v = \frac{c_1}{x} \]
\[ u = \int v = c_1 \ln x + c_2 \]

\[ \Rightarrow \quad y_h = x^{m_1} [c_1 \ln x + c_2]. \]

Case 3. \( \triangle < 0 \Rightarrow \) roots are complex conjugates, \( m_1 = a + bi \) and \( m_2 = a - bi \).

Thus

\[ y_h = c_1 x^{m_1} + c_2 x^{m_2} = c_1 x^{a+bi} + c_2 x^{a-bi} \]

or

\[ y_h = x^a [c_1 x^{bi} + c_2 x^{-bi}]. \]

Now

\[ x^{a+bi} = e^{(a+ib)\ln x} \text{ for } x > 0. \]

\[ \Rightarrow \]

\[ x^{bi} = e^{ib\ln x} = \cos(b \ln x) + i \sin(b \ln x) \]

\[ \Rightarrow \]

\[ y_h = x^a [A \cos(b \ln x) + B \sin(b \ln x)]. \]

**Example**  
Solve

\[ x^2 y'' + 7xy' + 5y = 0 \]

Here \( p = 7 \) and \( q = 5 \). The indicial equation (3) is for this example

\[ m^2 + m(p-1) + q = m^2 + 6m + 5 = (m+5)(m+1) \]

\[ \Rightarrow m = -5 \text{ or } -1 \Rightarrow \]

\[ y = c_1 x^{-5} + c_2 x^{-1} = \frac{c_1}{x^5} + \frac{c_2}{x}. \]