Ma 530

Sturm-Liouville Problems

We shall now develop some results that will be useful in our study of partial differential equations. First we define a *self-adjoint* operator

$$L(y) = [p(x)y']' + q(x)y$$

where $p(x), p'(x)$, and $q(x)$ are continuous and $p(x) \neq 0$ in some interval $a \leq x \leq b$.

Remark: Self-adjoint operators have special properties. Every second order linear operator can be put in self-adjoint form by multiplying it by a suitable factor. Consider

$$M(y) = y'' + R(x)y' + Q(x)y$$

Multiplying $M(y)$ by

$$e^{\int R(x)dx}$$

leads to

$$\begin{aligned} e^{\int R(x)dx}M(y) &= y''e^{\int R(x)dx} + Re^{\int R(x)dx}y' + Qe^{\int R(x)dx}y \\ &= \left(e^{\int R(x)dx}y'\right)' + Qe^{\int R(x)dx}y \end{aligned}$$

If we let $p(x) = e^{\int R(x)dx}$ and $q(x) = Qe^{\int R(x)dx}$, we see that $e^{\int R(x)dx}M$ has the form of $L(y)$ above. In particular, the differential equation $M(y) = 0$ may be rewritten as the self-adjoint differential equation $L(y) = 0$.

We now consider the eigenvalue problem with unmixed boundary conditions

$$\begin{aligned} L(y) + \lambda w(x)y &= 0 \quad a \leq x \leq b \\ \alpha_1 y(a) + \beta_1 y'(a) &= 0 \quad \alpha_1^2 + \beta_1^2 \neq 0 \\ \alpha_2 y(b) + \beta_2 y'(b) &= 0 \quad \alpha_2^2 + \beta_2^2 \neq 0 \end{aligned} \qquad (*)$$

where $w(x) \geq 0$ is a continuous function and $w(x)$ is not identically zero on $[a,b]$. $(*)$ is called a Sturm-Liouville problem. λ is a parameter independent of x. Note that the solution $y \equiv 0$ exists for all values of the parameter λ. It may be shown that nontrivial solutions exist for certain values of λ and not for other values of λ. If a nontrivial solution exists for a value $\lambda = \lambda_i$, then this value is called an *eigenvalue* of the operator L (relevant to the boundary conditions) and the corresponding nontrivial solution $y_i(x)$ is called an *eigenfunction.*

Definition: The *inner product* of two continuous functions $f(x)$ and $g(x)$ in the interval $[a,b]$ with respect to the weight function $w(x)$ is defined by

$$< f,g >_w = \int_a^b f(x)g(x)w(x)dx$$

(Here again we assume $w(x) \geq 0$ on $[a,b]$ and $w(x)$ is not identically zero in $[a,b]$.)

The inner product $< f,g >_w$ has the following properties:

$$< f,f >_w \; > 0$$
$$< f,f >_w \; = 0 \quad \Leftrightarrow f = 0$$
$$< f,g >_w \; =< g,f >_w$$
$$< \alpha f + \beta g, h >_w \; = \alpha < f,h >_w \; + \beta < g,h >_w \quad \alpha, \beta \text{ constants}$$

Definition. Two functions f and g are said to be orthogonal on $[a,b]$ with respect to the weight function $w(x)$ if

$$< f,g >_w \; = 0.$$

Example. Let $w(x) = 1$, then

$$\int_0^\pi \sin x \cos x dx = \frac{\sin^2 x}{2} |_0^\pi = 0$$

Therefore $\sin x$ and $\cos x$ are orthogonal on $[0,\pi]$ with respect to the weight function 1.

Definition. The set of continuous functions $\{f_1, f_2, \ldots\}$ is called an orthogonal set on $[a,b]$ with respect to the weight function $w(x)$ if

$$< f_m, f_n >_w \; = 0 \qquad m \neq n.$$

Example. $\left\{1, \cos \frac{\pi x}{L}, \cos \frac{2\pi x}{L}, \ldots, \cos \frac{n\pi x}{L}, \ \ldots\right\}$ is an orthogonal set on $[0,L]$ with respect to the weight function $w = 1$. For $m \neq n$ we have

$$< \cos \frac{m\pi x}{L}, \cos \frac{n\pi x}{L} >= \int_0^L \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = L\left[\frac{m \sin \pi m \cos \pi n - n \cos \pi m \sin \pi n}{\pi(m^2 - n^2)} \right] = 0$$

Remark. For vectors we have the following: if $\vec{u} = (u_1, \ldots, u_n)$ then the length of $\vec{u} = \|\vec{u}\| = (\sum u_i^2)^{\frac{1}{2}} = \sqrt{\vec{u} \cdot \vec{u}}$. Motivated by this we have the following definition.

Definition. Let $f(x)$ be a continuous function on $a \leq x \leq b$. Then the norm of f with respect to the weight function $w(x)$ is defined by

$$\|f\|_w = \sqrt{< f,f >_w} = \sqrt{\int_a^b f^2(x) w(x) dx}\,.$$

Example. $0 \leq x \leq 1, w(x) = 1$

$$\|x^2\|^2 =< x^2, x^2 >= \int_0^1 x^4 dx = \frac{x^5}{5} |_0^1 = \frac{1}{5}$$

$\Rightarrow$

$$\|x^2\| = \frac{1}{\sqrt{5}}$$

.

Remark. Let $y = \frac{x^2}{\|x^2\|} = \frac{x^2}{\sqrt{5}} \Rightarrow \|y\| = \frac{\|x^2\|}{\sqrt{5}} = 1.$

Definition. If $\|f\|_w = 1$, then f is said to be normalized.

Definition. A set of functions $\{\phi_1, \phi_2, \ldots\}$ defined on $[a, b]$ is called orthonormal if
(1) the set is orthogonal on $[a.b]$, and
(2) each function has norm 1 with respect to the weight function $w(x)$. Therefore $\{\phi_1, \phi_2, \ldots\}$ is an orthonormal set $\Leftrightarrow$

$$< \phi_i, \phi_j >_w \;= \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Example. $\{\sin(nx)\} = \{\sin x, \sin 2x, \sin 3x, \ldots\}$ on $[0, \pi]$ is an orthogonal set with respect to the weight function $w(x) = 1$ since

$$< \sin(mx), \sin(nx) >_1 \;= \int_0^{\pi} \sin mx \sin nx \, dx$$
$$= \frac{1}{2}\int_0^{\pi} [\cos(m-n)x - \cos(m+n)x] dx \qquad m \neq n$$
$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m-n}\right]_0^{\pi} = \frac{1}{2}\left[\frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)\pi}{m+n}\right] = 0$$

since m and n are integers.
Now

$$< \sin nx, \sin nx >_1 \;= \int_0^{\pi} \sin^2 nx dx$$
$$= \frac{1}{2}\int_0^{\pi}(1 - \cos 2nx) dx$$
$$= \frac{1}{2}\left(x - \frac{\sin 2nx}{2n}\right)\Big|_0^{\pi} = \frac{\pi}{2}.$$

Therefore

$$\|\sin nx\|_1 = \sqrt{< \sin nx, \sin nx >_1} = \sqrt{\frac{\pi}{2}}$$

$\Rightarrow$ this set is not orthonormal. We can make an orthonormal set from these functions by dividing each element in the original set by $\sqrt{\frac{\pi}{2}} \;\Rightarrow\; \left\{\sqrt{\frac{2}{\pi}} \sin nx\right\}$ is orthonormal set ($n = 1, 2, \ldots$).
We now present some results related to the eigenvalues and eigenfunctions of the self-adjoint BVP $(*)$.

Theorem 1: Eigenfunctions of $(*)$ corresponding to distinct eigenvalues are orthogonal with respect to the weight function $w(x)$ on $[a, b]$.

Remark: Theorem 1 may be extended to cover a number of cases which are important in applications.

1. Periodic Boundary Conditions:

Consider the BVP $(*)$ with the boundary conditions replaced by the periodic (mixed) boundary conditions

$$y(a) = y(b) \quad y'(a) = y'(b)$$

If we also assume that $p(a) = p(b)$, then Theorem 1 holds for this case also.

2. Singular End Points:

If in the Sturm-Liouville problem $(*)$ we have $p(a) = 0$, then we require that the solution $y(x)$ be bounded at $x = a$. Theorem 1 will then hold. Thus the requirement that $y(a)$ be finite replaces the first boundary condition in $(*)$ at $x = a$. Similar remarks hold if $p(b) = 0$. If both $p(a) = 0$ and $p(b) = 0$, then both boundary conditions in $(*)$ can be omitted and replaced by the requirements that the solution $y(x)$ be bounded at $x = a$ and $x = b$, and the conclusion of Theorem 1 still holds.

Fourier Series

Definition: A set of orthogonal functions $\{\psi_n(x)\}$ on an interval $[a,b]$ with respect to the weight function $w(x)$ is called *complete* if

$$\int_a^b w(x)f(x)\psi_n(x)dx = 0 \quad n = 1,2,3,\ldots$$

implies that $f = 0$ on $[a,b]$.

Theorem 2: If $f(x)$ is any continuous function on $[a,b]$, then $f(x)$ can be expanded at point in $a < x < b$ in a uniformly convergent Fourier series as

$$f(x) = \sum_{n=1}^{\infty} a_n\psi_n(x)$$

where

$$a_n = \frac{< f,\psi_n >_w}{< \psi_n,\psi_n >_w} = \frac{\int_a^b w(x)f(x)\psi_n(x)dx}{\int_a^b w(x)\psi_n^2(x)dx} \quad n = 1,2,\ldots \tag{1}$$

Here the set of functions $\{\psi_n\}$ is a complete, orthogonal set.

Proof: We shall establish the formula for a_n. Now

$$< \psi_k,f(x) >_w \; =< \psi_k,\sum_1^{\infty} a_n\psi_n >_w$$

$$=< \psi_k,a_1\psi_1 + a_2\psi_2 + \cdots >_w$$

$$= a_1 < \psi_k,\psi_1 >_w + a_2 < \psi_k,\psi_2 >_w + \cdots + a_k < \psi_k,\psi_k >_w + a_{k+1} < \psi_k,\psi_{k+1} >_w + \cdots$$

But $< \psi_k,\psi_j >_w = 0$ if $j \neq k$, since the set $\{\psi_k\}$ is orthogonal.

$\Rightarrow$

$$< \psi_k,f(x) >_w = a_k < \psi_k,\psi_k >_w = a_k\|\psi_k\|_w^2$$

This is (1). (1) is the formula for the coefficients in the expansion of a function $f(x)$ in terms of a set of orthogonal functions.

Ordinary Fourier Series

Fourier Sine Series

Consider the eigenvalue problem

$$D.E.\quad y'' + \lambda y = 0 \qquad 0 \le x \le L \qquad B.C.\ \ y(0) = y(L) = 0 \qquad \lambda > 0$$

We shall first solve this problem. Now the general solution to the DE is

$$y = c_1 \sin\sqrt{\lambda}\,x + c_2 \cos\sqrt{\lambda}\,x.$$

$$\begin{aligned}
&y(0) = 0 \Rightarrow y = c_1 \cdot 0 + c_2 = 0 \Rightarrow c_2 = 0\\
&y(L) = 0 \Rightarrow c_1 \sin\sqrt{\lambda}\,L = 0 \Rightarrow c_1 = 0 \text{ or } \sin\sqrt{\lambda}\,L = 0\\
&\sin\sqrt{\lambda}\,L = 0 \Rightarrow \sqrt{\lambda}\,L = n\pi \qquad n = 1,2,\ldots
\end{aligned}$$

$\Rightarrow$

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

are the eigenvalues, whereas the eigenfunctions are

$$\sin\sqrt{\lambda_n}\,x = \sin\frac{n\pi}{L}x = \psi_n$$

an orthogonal set.

To make $\left\{\sin\frac{n\pi x}{L}\right\}$ orthonormal set we divide each function by $\left\|\sin\frac{n\pi x}{L}\right\| = \sqrt{\frac{L}{2}}$. Therefore $\left\{\sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}\right\}$ is an orthonormal set.

Hence if

$$f(x) = \sum_{1}^{\infty} \alpha_k \sin\frac{k\pi x}{L}$$

then from (1) above

$$\alpha_k = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}dx,$$

since

$$\int_0^L [\psi_k(x)]^2 dx = \int_0^L \sin^2\left(\frac{n\pi}{L}\right)x\,dx = \frac{L}{2}.$$

These formulas are for the Fourier *sine* series for $f(x)$ on $0 < x < L$.

Remarks. 1. At $x = 0$ and $x = L$ $\quad\sum \alpha_k \sin\frac{k\pi x}{L}$ gives 0 for $f(x)$. Therefore unless $f(0) = f(L) = 0$ the Fourier series is not good at the end points.

2. Since $\sin\frac{k\pi}{L}(x+2L) = \sin\left(\frac{k\pi}{L}x + 2k\pi\right) = \sin\frac{k\pi x}{L}$, we see that the Fourier series yields $f(x+2L) = f(x) \Rightarrow$ Fourier series has period $2L$. For $-L < x < 0$

we have $\sum_1^\infty \alpha_k \sin \frac{k\pi x}{L} = \sum_1^\infty \alpha_k \sin\left(\frac{-k\pi(-x)}{L} \right)$

$$= -\sum_1^\infty \alpha_k \sin \frac{k\pi(-x)}{L} \qquad -L < x < 0 \Rightarrow L > -x > 0$$

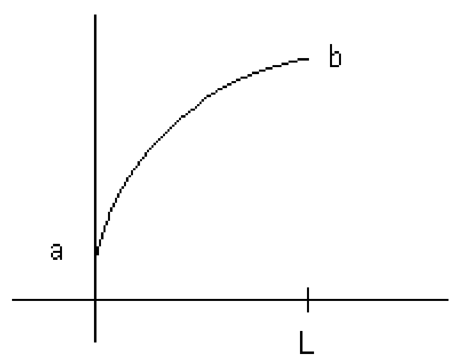
$$= -f(-x), \text{ where } f(x) \text{ is value of series in } 0 < x < L.$$

Therefore the Fourier sine series converges to function $F(x)$ where

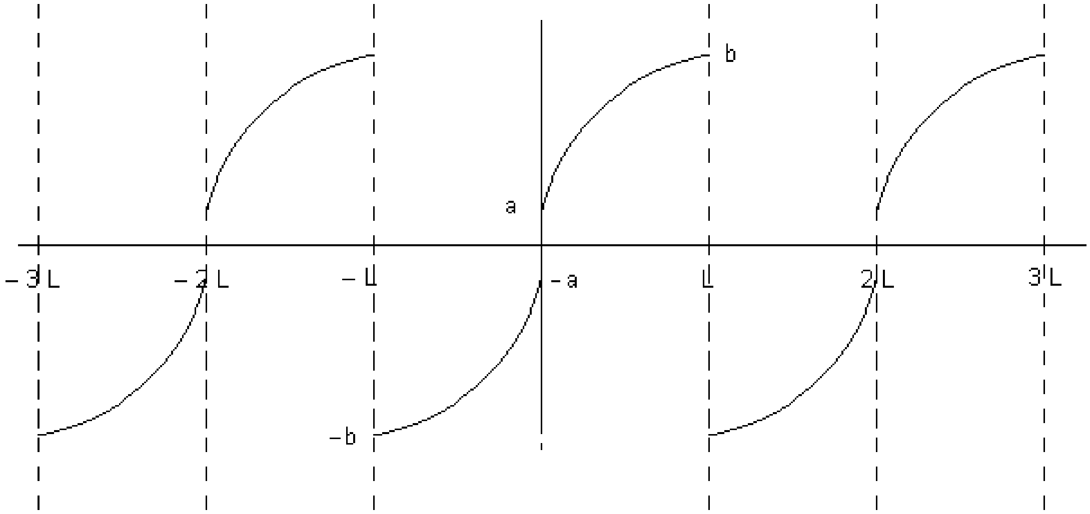
$$F(x) = \begin{cases} f(x) & 0 < x < L \\ -f(-x) & -L < x < 0 \end{cases} \qquad F(x + 2L) = F(x)$$

This is the odd periodic extension of $f(x)$ with period $2L$. Unless $f(\pm kL) = 0$ $F(x)$ will be discontinuous at $\pm L, \pm 2L, \ldots$ Note that the function $f(x)$ is given on $[0, L]$ only, whereas the Fourier Sine series extends it to a function $F(x)$ which is defined on $-\infty < x < \infty$.

Suppose that the graph of the function $f(x)$ is given by the figure below.

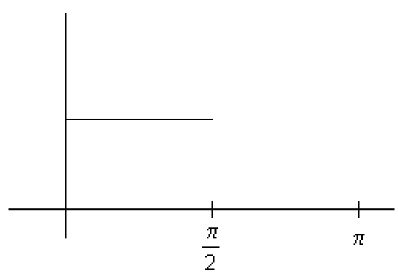


Then the Fourier sine series generates a function $F(x)$ defined on $-\infty < x < \infty$ whose graph is given below.



Example Find the Fourier sine series of

$$f(x) = \begin{cases} 1 & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$



Now

$$f(x) = \sum \alpha_n \sin \frac{n\pi x}{L} = \sum_{1}^{\infty} \alpha_n \sin nx,$$

since $2L = 2\pi \Rightarrow L = \pi$.
The formula above for the coefficients in the Fourier sine series implies

$$\alpha_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx = \frac{2}{\pi}\int_0^\pi f(x) \sin nx dx$$

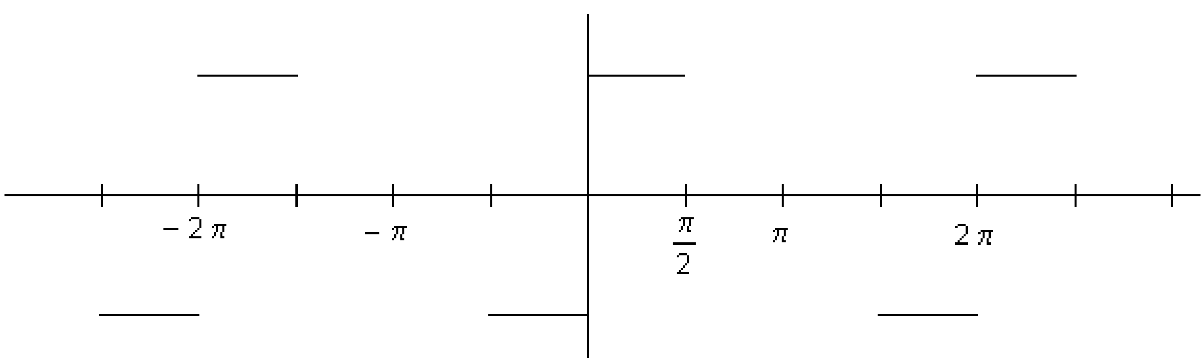
$$\alpha_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 1 \cdot \sin nx dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} 0 \cdot \sin nx dx = -\frac{2}{\pi}\frac{\cos nx}{n}\Big|_0^{\frac{\pi}{2}}$$
$$= -\frac{2}{\pi n}\left[\cos\frac{n\pi}{2} - 1\right]$$

$$\alpha_n = \begin{cases} \frac{2}{\pi n} & n \text{ odd} \\ \left(\frac{-2}{\pi n}\right)\left[(-1)^{\frac{n}{2}} - 1\right] & n \text{ even} \end{cases}$$

Therefore

$$f(x) = \sum_1^{\infty} \alpha_n \sin nx$$
$$= \frac{2}{\pi}\left[\sin x + \frac{2}{2}\sin 2x + \frac{1}{3}\sin 3x + 0 \cdot \sin 4x + \frac{1}{5}\sin 5x + \frac{2}{6}\sin 6x + \cdots\right]$$

Note that our function $f(x)$ on $0 \le x \le \pi$ is extended to the following on $-\infty < x < \infty$.



What we have done with *sine* functions can be done with *cosine* functions.

Fourier Cosine Series.

This comes from eigenvalue problem

$$\text{D.E.} \quad y'' + \lambda y = 0 \qquad \text{B.C.} \quad y'(0) = y'(L) = 0$$

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

are the eigenvalues and

$$\psi_n = \cos\frac{2\pi x}{L}$$

are the eigenfunctions, $n = 0, 1, 2, \ldots$.

Note $\lambda_0 = 0 \Rightarrow \psi_0 = 1$ which is an eigenfunction. Now we want to write

$$f(x) = \beta_0 + \sum_1^{\infty} \beta_n \cos\frac{n\pi x}{L}$$

Proceeding as above in our derivation of the constants in the Fourier Sine series, we get for the constants in the Fourier Cosine series

$$\beta_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \qquad \beta_0 = \frac{1}{L}\int_0^L f(x)dx$$

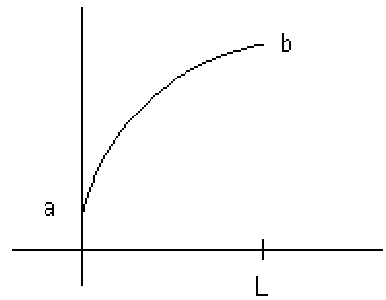
To see where the formula for β_0 comes from note

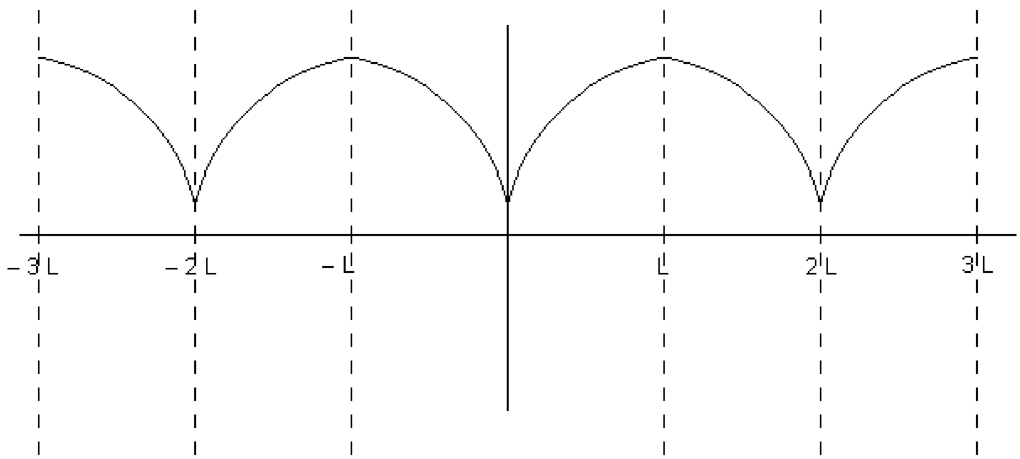
$$< \psi_0, f(x) >_1 \; =< \psi_0, \beta_0\psi_0 >_1 \; =< 1, 1 >_1 \; \beta_0$$

$$\Rightarrow \beta_0 = \frac{\int_0^L 1 \cdot f(x)dx}{\int_0^L 1^2 dx} = \frac{1}{L}\int_0^L f(x)dx.$$

Again the Fourier series is periodic with period $2L$. However, now $f(-x) = f(x)$ since *cosine* is an even function. Here the Fourier Cosine series extends $f(x)$ which is given on $[0, L]$ to a function $F(x)$ which is defined on $-\infty < x < \infty$ as

$$F(x) = \begin{cases} f(x) & 0 < x < L \\ f(-x) & -L < x < 0 \end{cases} \qquad F(x + 2L) = F(x).$$





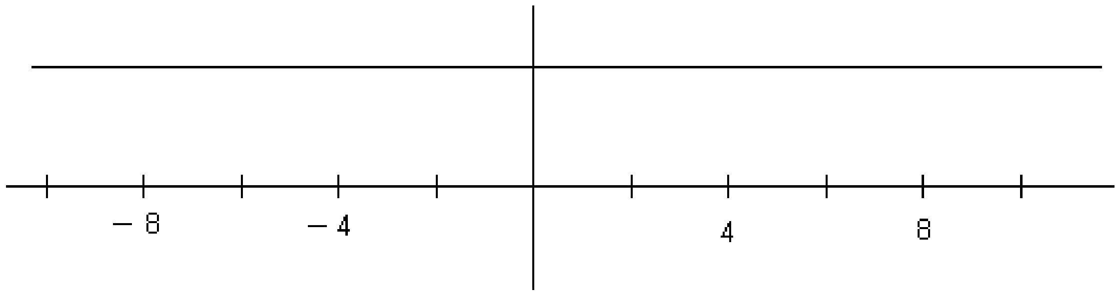
Example. Find the Fourier Cosine series for $f(x) = 1, \qquad 0 < x < 4$

$L = 4$

$$f(x) = \beta_0 + \sum_1^{\infty} \beta_n \cos \frac{n\pi x}{4} \qquad \beta_0 = \tfrac{1}{4}\int_0^4 f(x)dx = \tfrac{1}{4}\int_0^4 1 \cdot dx = 1$$

$$\beta_k = \tfrac{2}{4}\int_0^4 1 \cdot \cos \frac{n\pi x}{4} dx = \tfrac{1}{2}\left[\frac{\sin \frac{n\pi x}{4}}{\frac{n\pi}{4}} \right]_0^4 = \frac{2}{4\pi n}[\sin 0] = 0$$

Therefore $f(x) = 1$ is its own Fourier Cosine series. The function is simply extended.



Full Fourier Sine and Cosine Series

This comes from the eigenvalue problem

$$D.E.\ \ y'' + \lambda y = 0 \qquad B.C.\, y(0) = y(2L) \qquad y'(0) = y'(2L) \qquad 0 \le x \le 2L$$

The eigenvalues are

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

$n = 0, 1, 2, \ldots,$, whereas the eigenfunctions are

$$\psi_n = a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{2\pi x}{L} \qquad n = 0, 1, 2, ..$$

.

Note that for this problem the function $f(x)$ is given on $[0, 2L]$ since the eigenvalue problem is given on this interval. This is a different interval than that for Fourier Sine and Fourier Cosine series.

$\Rightarrow$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{2L}\int_0^{2L} f(x)dx, \qquad a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}dx \qquad b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}dx$$

Example Find full Fourier series for

$$f(x) = \begin{cases} 1 & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

$$2L = \pi \Rightarrow L = \frac{\pi}{2}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 1 \cdot dx = \frac{1}{2}$$

$$a_n = \frac{1}{\frac{\pi}{2}}\int_0^{\pi} f(x)\cos\frac{n\pi x}{\frac{\pi}{2}}dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 1 \cdot \cos 2nx\, dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos 2nx\, dx = \frac{2}{\pi}\frac{\sin 2nx}{2n}\Big|_0^{\frac{\pi}{2}} = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin 2nx\, dx = -\frac{2}{\pi}\frac{\cos 2nx}{2n}\Big|_0^{\frac{\pi}{2}} = \frac{1}{\pi n}[\cos n\pi - \cos 0] \qquad n = 1,2,\ldots$$

$$b_n = -\frac{1}{\pi n}[(-1)^n - 1] = \begin{cases} +\frac{2}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\sin 2x + \frac{1}{3}\sin 6x + \frac{1}{5}\sin 10x + \cdots\right]$$

Fourier-Bessel Series

An important boundary value problem involving Bessel's equation is

$$x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0 \qquad 0 \le x \le c$$

$$y(c) = 0 \quad y(0) \text{ is finite}$$

where n is a nonnegative integer. The differential equation can be put in the self-adjoint form

$$(xy')' + \frac{(\lambda^2x^2 - n^2)}{x}y = 0$$

This is a special case of the Sturm-Liouville problem $(*)$ with $p(x) = x$ and the weight function $w(x) = x.$ Since $p(0) = 0,$ no boundary condition is required at $x = 0;$ the condition that $y(0)$ be finite can be considered as a hidden boundary condition.

The general solution of the differential equation can be written in terms of the Bessel functions of order n of the first and second kind, namely,

$$y = AJ_n(\lambda x) + BY_n(\lambda x)$$

Since $Y_n(x)$ is unbounded at the origin, we set $B = 0.$ Also, we have

$$y(c) = J_n(\lambda c) = 0$$

If we let $\alpha_i,\ i = 1,2,\ldots$ denote the positive roots of $J_n(\alpha) = 0,$ then the eigenvalues are

$$\lambda c = \alpha_i \ \text{ or } \ \lambda = \lambda_i = \frac{\alpha_i}{c} \quad i = 1,2,\ldots$$

with corresponding eigenfunctions

$$J_n(\lambda_i x) \quad i = 1,2,\ldots$$

The set of functions $\{J_n(\lambda_i x)\}$ for a fixed n form a complete, orthogonal set on $[0,c]$ with respect to the weight function x. Therefore, from the expression that we derived above for the coefficients in a Fourier expansion in terms of orthogonal functions we have

$$f(x) = \sum_{i=1}^{\infty} a_i J_n(\lambda_i x)$$

where the coefficients a_i are given by

$$a_i = \frac{\int_0^c x J_n(\lambda_i x) f(x)dx}{\int_0^c x[J_n(\lambda_i x)]^2 dx} \quad i = 1, 2, \ldots$$

It can be shown that

$$\int_0^c x[J_n(\lambda_i x)]^2 dx = \frac{c^2}{2}[J_{n+1}(\lambda_i c)]^2$$

so that

$$a_i = \frac{2\int_0^c x J_n(\lambda_i x) f(x)dx}{c^2[J_{n+1}(\lambda_i c)]^2} \quad i = 1, 2, \ldots$$

Example: Expand the function $f(x) = 1$ into a Fourier-Bessel series of the zeroth order in $0 < x < 1$.

Solution: The coefficients a_i are given by

$$a_i = \frac{2\int_0^1 x J_0(\lambda_i x)dx}{[J_1(\lambda_i c)]^2} \quad i = 1, 2, \ldots$$

where the λ_i are the roots of $J_0(\lambda_i) = 0$. Since

$$\frac{d}{dx}[xJ_x(x)] = xJ_0(x)$$

then

$$\int_0^x tJ_0(t)dt = xJ_1(x)$$

Letting $\lambda_i x = t$ we have

$$\int_0^1 xJ_0(\lambda_i x)dx = \int_0^{\lambda_i} \frac{t}{\lambda_i} J_0(t)\frac{dt}{\lambda_i} = \frac{1}{\lambda_i^2}[\lambda_i J_1(\lambda_i)] = \frac{J_1(\lambda_i)}{\lambda_i}$$

Thus

$$a_i = \frac{2}{\lambda_i J_1(\lambda_i)}$$

so that

$$f(x) = 2\sum_{i=1}^{\infty} \frac{J_0(\lambda_i x)}{\lambda_i J_1(\lambda_i)}$$