Ma 530 Method of Frobenius - Some Examples

Example 1

Use the method of series solution near a regular singular point to find a Frobenius solution to

$$x^2y'' + xy' + \left(x^2 - 4\right)y = 0$$

and show that it can be written as

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n-1}(n+2)!n!}$$

Solution:

$$y'' + \frac{1}{x}y' + \frac{\left(x^2 - 4\right)}{x^2}y = 0$$

Obviously $x = 0$ is a singular point, since

$$P(x) = \frac{1}{x}$$

$$Q(x) = \frac{x^2 - 4}{x^2}$$

$$xP(x) = 1$$

$$x^2Q(x) = x^2 - 4$$

so we have a regular singular point at $x = 0$

$$p_0 = \lim_{x\to 0} xP(x) = 1$$

$$q_0 = \lim_{x\to 0} x^2Q(x) = -4$$

Thus the indicial equation is

$$m^2 - 4 = 0$$

So the larger root is $m = 2$, and the series is $y = x^2 \sum_{n=0}^{\infty} a_n x^n$

$$y = x^2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$y' = \sum_{n=0}^{\infty} a_n (2+n) x^{n+1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (2+n)(n+1) x^n$$

$$x^2\sum_{n=0}^{\infty}a_n(2+n)(n+1)x^n + x\sum_{n=0}^{\infty}a_n(2+n)x^{n+1} + \left(x^2-4\right)\sum_{n=0}^{\infty}a_nx^{n+2} = 0$$

$$\sum_{n=0}^{\infty}a_n(2+n)(n+1)x^{n+2} + \sum_{n=0}^{\infty}a_n(2+n)x^{n+2} + \sum_{n=0}^{\infty}a_nx^{n+4} - 4\sum_{n=0}^{\infty}a_nx^{n+2} = 0$$

$$\sum_{n=0}^{\infty}a_n(2+n)(n+1)x^{n+2} + \sum_{n=0}^{\infty}a_n(2+n)x^{n+2} + \sum_{n=2}^{\infty}a_{n-2}x^{n+2} - 4\sum_{n=0}^{\infty}a_nx^{n+2} = 0$$

$$5a_1x^3 + \sum_{n=2}^{\infty}(a_n(2+n)(n+1) + a_n(2+n) + a_{n-2} - 4a_n)x^{2+n} = 0$$

$$5a_1x^3 + \sum_{n=2}^{\infty}\left(4a_nn + a_nn^2 + a_{n-2}\right)x^{2+n} = 0$$

Thus

$$a_1 = 0$$

and the recurrence relation is

$$a_n(n)(n+4) + a_{n-2} = 0 \quad n = 2,3,\ldots$$

or

$$a_n = -\frac{a_{n-2}}{n(n+4)} \quad n = 2,3,\ldots$$

$$a_2 = a_{2(1)} = -\frac{a_0}{2(6)} = -\frac{a_0}{2^1(3!)}$$

$$a_3 = 0$$

$$a_4 = a_{2(2)} = -\frac{a_2}{(4)(8)} = +\frac{a_0}{2(4)(6)(8)} = \frac{a_0}{2^4(2)(3)(4)} = \frac{a_0}{2^3(2!)(4!)}$$

$$a_5 = 0$$

$$a_6 = a_{2(3)} = -\frac{a_4}{6(10)} = -\frac{a_0}{2^2(3)(5)\left(2^3\right)(2!)(4!)} = -\frac{a_0}{2^5(3!)(5!)}$$

Therefore $a_{2m+1} = 0, n = 0,1,2,\ldots$ and

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m-1}(m!)(m+2)!}$$

and this is the result.

Example 2

Find a series solution about the regular singular point $x = 0$ of

$$x^2y'' - xy' + (1-x)y = 0$$

We rewrite the equation as

$$y'' - \frac{1}{x}y' + \frac{1-x}{x^2}y = 0$$

Then

$$p_0 = \lim_{x\to 0} xP(x) = \lim_{x\to 0} x\left(-\frac{1}{x}\right) = -1$$

$$q_0 = \lim_{x\to 0} x^2 Q(x) = \lim_{x\to 0} x^2\left(\frac{1-x}{x^2}\right) = 1$$

The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 + (-1-1)r + 1 = r^2 - 2r + 1 = (r-1)^2 = 0$$

Thus $r_1 = r_2 = 1$ and

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+1}$$

The DE implies

$$\sum_{n=0}^{\infty} a_n (n+1)(n) x^{n+1} - \sum_{n=0}^{\infty} a_n (n+1) x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Shifting the last sum above by letting $k+1 = n+2$, or $n = k-1$, we have

$$+\sum_{n=0}^{\infty} a_n \{n(n+1) - (n+1) + 1\} x^{n+1} - \sum_{k=1}^{\infty} a_{k-1} x^{k+1} = 0$$

Simplifying the first sum we have

$$\sum_{m=1}^{\infty} \left[m^2 a_m - a_{m-1}\right] x^{m+1} = 0$$

Thus

$$a_m = \frac{1}{m^2} a_{m-1} \qquad m \geq 1$$

For $m = 1, 2,$ and 3 we have

$$a_1 = a_0$$

$$a_2 = \frac{1}{2^2} a_1 = \frac{1}{(2 \cdot 1)^2} a_0$$

$$a_3 = \frac{1}{3^2} a_2 = \frac{1}{(3 \cdot 2 \cdot 1)^2} a_0$$

In general

$$a_m = \frac{1}{(m!)^2} a_0$$

and

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+1} = a_0 \sum_{m=0}^{\infty} \frac{1}{(m!)^2} x^{m+1}$$

Example 3

Now find the first few terms of the second series solution about the regular singular point $x = 0$ of

$$x^2 y'' - xy' + (1-x)y = 0$$

Since we have equal roots of the indicial equation

$$y_2 = y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+1}$$

Thus

$$y_2' = y_1' \ln x + x^{-1} y_1 + \sum_{0}^{\infty} b_n (n+1) x^n$$

$$y_2'' = y_1'' \ln x - x^{-2} y_1 + 2x^{-1} y_1 + \sum_{1}^{\infty} b_n (n+1)(n) x^{n-1}$$

Substituting into the DE and collecting terms we have

$$\left\{ x^2 y_1'' - x y_1' + (1-x) y_1 \right\} \ln x - 2y_1 + 2x y_1'$$
$$+ \sum_{n=1}^{\infty} b_n (n+1)(n) x^{n+1} - \sum_{n=0}^{\infty} b_n (n+1) x^{n+1} + \sum_{n=0}^{\infty} b_n x^{n+1} - \sum_{n=0}^{\infty} b_n x^{n+2}$$
$$= 0$$

Since y_1 is a solution of the DE, the terms times $\ln x$ above equal 0, and we have

$$2x y_1' - 2y_1 + \sum_{n=1}^{\infty} b_n (n+1)(n) x^{n+1} - \sum_{n=0}^{\infty} b_n (n+1) x^{n+1} + \sum_{n=0}^{\infty} b_n x^{n+1} - \sum_{n=0}^{\infty} b_n x^{n+2} = 0$$

Combining the first 3 sums and shifting the last one by letting $n = k-1$, leads to

$$2x y_1' - 2y_1 + \sum_{1}^{\infty} b_n \left(n^2\right) x^{n+1} - \sum_{k=2}^{\infty} b_{k-1} x^{k+1} = 0$$

$$2x y_1' - 2y_1 + b_1 x^2 + \sum_{m=2}^{\infty} \left(m^2 b_m - b_{m-1}\right) x^{m+1} = 0$$

$$y_1' = \sum_{n=0}^{\infty} \frac{(n+1)}{(n!)^2} x^n$$

Thus substituting y_1 and y_1' into the above we have

$$\sum_{n=0}^{\infty} \frac{2(n+1)-2}{(n!)^2} x^{n+1} + b_1 x^2 + \sum_{m=2}^{\infty} \left(m^2 b_m - b_{m-1}\right) x^{m+1} = 0$$

or

$$(2+b_1) x^2 + \sum_{m=2}^{\infty} \left[\frac{2m}{(m!)^2} + m^2 b_m - b_{m-1} \right] x^{m+1} = 0$$

We now set the coefficients of the powers of x equal to 0.

$$b_1 = -2$$

$$\frac{2m}{(m!)^2} + m^2 b_m - b_{m-1} = 0, \quad m \geq 2$$

or

$$b_m = \frac{1}{m^2} \left[b_{m-1} - \frac{2m}{(m!)^2} \right] \quad m \geq 2$$

For $m = 2$ and 3 we get

$$b_2 = \frac{1}{2^2}[b_1 - 1] = -\frac{3}{4}$$

$$b_3 = \frac{1}{3^2}\left[-\frac{3}{4} - \frac{6}{36}\right] = -\frac{11}{108}$$

Thus

$$y_2 = y_1 \ln x + \sum_{n=1}^{\infty} b_n x^{n+1} = y_2 = y_1 \ln x - 2x^2 - \frac{3}{4}x^3 - \frac{11}{108}x^4 + \cdots$$