Ma 530

Infinite Series I

Please note that in addition to the material below this lecture incorporated material from the Visual Calculus web site. The material on sequences is at Visual Sequences. (To use this link hold down the Ctrl key and click.)

Definitions

Definition A series is a sequence of terms that you intend to add up.

A finite series has a finite number of terms and the sum is well-defined and independent of the order in which the terms are added:

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

An infinite series has an infinite number of terms and hence the sum is not necessarily well-defined and may in fact depend on the order in which the terms are added. Whether or not the sum is well-defined we still write the series as:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

The initial index may or may not be 1. If we don't say whether a series is finite or infinite, we normally mean an ***infinite*** series.

We will later give a precise way to add an infinite series, but we first give an example of the problems that can arise if you add an infinite series incorrectly:

Example What is wrong with the following proof that $0 = 1$?

$$\begin{aligned} 0 &= \quad 0 \quad + \quad 0 \quad + \quad 0 \quad + \cdots \\ &= (1-1) + (1-1) + (1-1) + \cdots \\ &= 1 + (-1+1) + (-1+1) + (-1+1) + \cdots \\ &= 1 + \quad 0 \quad + \quad 0 \quad + \quad 0 \quad + \cdots \\ &= 1 \end{aligned}$$

Solution: The associative rule does not work for an infinite sum.

Remark: Thus if we add the terms of the sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ which is called a series and is denoted by $\sum_{n=1}^{\infty} a_n$.

Definition — The Sum of the Series

Definition Given an infinite series $S = \sum_{n=1}^{\infty} a_n$, its k^{th}-partial sum is the finite series

$S_k = \sum_{n=1}^{k} a_n$. Then, the sum of the infinite series is defined to be

$$S = \lim_{k\to\infty} S_k \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} \sum_{n=1}^{k} a_n$$

provided the limit exists or is positive or negative infinity.

Remark This says that an infinite sum ***must*** be computed in the order the terms are listed in the series:

$$\sum_{n=1}^{\infty} a_n = (\cdots\{[(a_1 + a_2) + a_3] + a_4\} + \cdots)$$

Remark Thus, an infinite series is associated with two sequences:

- the sequence of terms: $\{a_n\}$ and
- the sequence of partial sums: $\{S_k\}$.

The sum of the series (or simply the series) is the sum of the sequence of terms and is the limit of the sequence of partial sums.

Further Terminology:

- If the limit exists (i.e. $S = \lim_{k\to\infty} S_k$ is finite), we say the series exists or is convergent or converges to S or has sum S.
- If the limit does not exist (i.e. $\lim_{k\to\infty} S_k$ does not exist), we say the series does not exist or is divergent or diverges or does not have a sum.
- If the limit is positive infinity (i.e. $\lim_{k\to\infty} S_k = \infty$), we say the series diverges to ∞.
- If the limit is negative infinity (i.e. $\lim_{k\to\infty} S_k = -\infty$), we say the series diverges to $-\infty$.

To say that the limit is positive or negative infinity does ***not*** say that the limit exists! It merely says the ***way*** in which it does not exist, i.e. the ***way*** in which it ***diverges***.

Geometric Series — Finite

Definition A geometric series is a series in which the ratio of successive terms is a constant.

We begin with finite geometric series:

Example The finite series $S = \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + \frac{3}{64} + \frac{3}{128}$ is geometric and the ratio of successive terms is $\frac{1}{2}$. The sum is $S = \frac{189}{128}$. This series may be written in summation notation in many ways such as:

$$S = \sum_{n=1}^{6} \frac{3}{2^{n+1}} = \sum_{n=0}^{5} \frac{3}{2^{n+2}} = \sum_{n=2}^{7} \frac{3}{2^n} = \sum_{n=0}^{5} \frac{3}{4}\left(\frac{1}{2}\right)^n$$

In any case, the first term is $\frac{3}{4}$, the ratio of successive terms is $\frac{1}{2}$, there are 6 terms and the sum is $\frac{189}{128}$.

We write the general finite geometric series as

$$S = \sum_{n=0}^{k} ar^n = a + ar + ar^2 + \cdots + ar^{k-1} + ar^k$$

although many other forms are possible. In this series, the first term is a, the ratio of successive terms is r and there are $k + 1$ terms.

Summing the Series

Gauss found a way to write down the general sum without using a summation ($\sum$) or an ellipsis ($\cdots$). Proceed as follows: Multiply the series by r:

$$rS = ar + ar^2 + ar^3 + \cdots + ar^k + ar^{k+1}$$

Subtract the formula for rS from the formula for S and notice that all terms cancel except the first term in S and the last term in rS :

$$S - rS = a - ar^{k+1}$$

If $r \neq 1$, this may be solved for S:

$$S = \frac{a(1 - r^{k+1})}{1 - r}$$

If $r = 1$, the original series may be easily summed:

$$S = \sum_{n=0}^{k} a = a + a + \cdots + a = (k + 1)a$$

since there are $k + 1$ terms each of which is a.

In summary, the general finite geometric series is

$$S = \sum_{n=0}^{k} ar^n = a + ar + ar^2 + \cdots + ar^{k-1} + ar^k$$

$$= \begin{cases} \dfrac{a(1 - r^{k+1})}{1 - r} & \text{if} \quad r \neq 1 \\ (k + 1)a & \text{if} \quad r = 1 \end{cases}$$

where the first term is a, the ratio of successive terms is r and the number of terms is $k + 1$.

Example For our original example $S = \sum_{n=1}^{6} \frac{3}{2^{n+1}}$, find the sum by using the general formula.

Solution: The first term is $\frac{3}{4}$, the ratio of successive terms is $\frac{1}{2}$ and there are 6 terms. So the sum is $S = \dfrac{\frac{3}{4}\left(1 - \left(\frac{1}{2}\right)^6\right)}{1 - \left(\frac{1}{2}\right)} = \dfrac{\frac{3}{4}\left(\frac{63}{64}\right)}{\frac{1}{2}} = \frac{189}{128}$. Notice that we do not need to write the summation with the index starting at 0 before identifying the first term, the ratio, or the number of terms.

Example $\sum_{n=2}^{\infty}\left(\frac{\pi}{4}\right)^n$ is a Geometric Series where $a = \frac{\pi^2}{16}$ and $r = \frac{\pi}{4}$. Its sum $= \dfrac{\frac{\pi^2}{16}}{1 - \frac{\pi}{4}} = -\frac{1}{4}\frac{\pi^2}{\pi - 4}$.

Your turn:

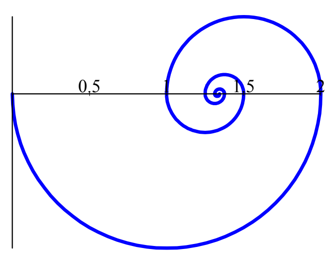
Exercise Compute $\sum_{p=2}^{7} \frac{2}{3^p}$ ....

Geometric Series — Applications

Exercise A ball is dropped from a height of 12 feet. Each time it bounces it reaches a height which is $\frac{2}{3}$ of the height on the previous bounce.

1. What is the total distance travelled by the ball (on the infinite number of bounces)?.. ... ...
2. What is the total time the ball takes to travel this distance?. ... ...

Exercise The spiral at the right is made from an infinite number of semicircles whose centers are all on the x-axis. The first semicircle is centered at $x = 1$ and has radius $r = 1$. The radius of each subsequent semicircle is half of the radius of the previous semicircle.



1. Consider the infinite sequence of points where the spiral crosses the x-axis. What is the x-coordinate of the limit of this sequence?....... ... ...
2. What is the total length of the spiral (with an infinite number of

semicircles)?

Or, is the length infinite?.. ... ...

Telescoping Series

Telescoping series are another class of series which can be summed exactly. Unfortunately, there is no precise definition of a telescoping series. Suffice it to say that: part of each term cancels with part of one or more subsequent terms allowing one to explicitly compute the partial sums and hence the total sum of the series. The best way to understand telescoping series is through examples.

Example Compute $S = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$.

Solution: To see what is happening, we first write out the first six terms in two ways. On the one hand, we combine the fractions:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)}\right) = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \cdots$$

In this form it is very hard to tell what the sum is. However, in the original form we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots$$

Part of each term cancels part of the next term. However, we cannot use the associative rule. So we need to look at the partial sums.

We compute the k^{th} partial sum and cancel everything except the first half of the first term and the last half of the last term:

$$\begin{aligned} S_k = \sum_{n=1}^{k}\left(\frac{1}{n} - \frac{1}{n+1}\right) &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= 1 - \frac{1}{k+1} \end{aligned}$$

So the sum of the series is

$$S = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(1 - \frac{1}{k+1}\right) = 1$$

Convergence and Divergence Tests

Why is it important to know when a series converges? Because, for example:

Convergent Series May be Used to Define and Approximate Fundamental Constants

Mathematicians often use series to compute decimal values for fundamental constants (like π and e) or to define new fundamental constants. Here are some examples.

Examples of Convergent Series Used to Define and Approximate Fundamental Constants

You are not yet expected to be able to prove the convergence of these series nor to estimate the error in the approximation.

Example Compute $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots$

Remark Recall that n factorial is $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$ and by definition $0! = 1$.

Solution: This series is neither geometric nor telescoping. So you do not yet know how to compute the sum. However, it can be shown to converge (by the Ratio Test). Taking 21 terms the partial sum is $S_{20} = \displaystyle\sum_{n=0}^{20} \frac{1}{n!} = 2.718\,281\,828\,459\,045\,235\,3$ which can be shown to be correct to within $\pm 10^{-19}$ (using the Taylor Bound on the Remainder). In fact, Taylor's theorem shows that the sum of the infinite series is e. So this is an excellent way to find a decimal approximation to e.

Example Compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots$

Solution: This series is called the p-series with $p = 3$. It can be shown to converge (by the Integral Test). Taking 25 terms the partial sum is $S_{25} = \displaystyle\sum_{n=1}^{25} \frac{1}{n^3} = 1.201\,29$ which can be shown to be correct to within $\pm 10^{-3}$ (using the Integral Bound on the Remainder). Riemann recognized that the sum of this infinite series is a new transcendental number (not expressible in terms of π and e) and named it $\zeta(3)$, which is read "zeta of 3".

Example Consider the series $S = 5 + 4 + 3 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ where later terms are each half of the preceding term. Is this series convergent or divergent and why?

Solution: If we ignore the first three terms, the tail is the geometric series $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ whose ratio is $\frac{1}{2}$. Consequently, the tail converges and so the series converges. In fact the sum of the geometric tail is $\dfrac{2}{1 - \frac{1}{2}} = 4$. So the sum of the original series is $S = 5 + 4 + 3 + (4) = 16$.

Definition A series $\displaystyle\sum_{n=n_o}^{\infty} a_n$ is positive if all of its terms are positive, $a_n > 0$ for all n.

A series $\displaystyle\sum_{n=n_o}^{\infty} a_n$ is negative if all of its terms are negative, $a_n < 0$ for all n.

A series $\displaystyle\sum_{n=n_o}^{\infty} a_n$ is indefinite if some terms are positive and some terms are negative.

With these examples in mind, we can now turn to the Convergence and Divergence Tests. There is one test which can show that a series is divergent. There are tests which can be used for series whose

terms are all positive. And there are tests which can be used for series whose terms are both positive and negative.
In addition, there is one general principle, used all the time, which we discuss now. It says that to determine the convergence of a series, we can ignore any number of initial terms and only look at the remaining part of the series, called the tail.

Definition A tail of the series $\sum_{n=i}^{\infty} a_n$ is any series of the form $\sum_{n=N}^{\infty} a_n$ where $N > i$.

Testing a Tail

A series $\sum_{n=n_o}^{\infty} a_n$ is convergent if and only if any (and hence every) tail is convergent.

Remark This says the convergence of a series does not depend on any finite number of terms. Further, you can check for convergence by applying a convergence test to the tail.

Example Consider the series $S = 5 + 4 + 3 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ where later terms are each half of the preceding term. Is this series convergent or divergent and why?

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n^{th} Term Divergence Test

Proposition ***The n^{th} Term Divergence Test***

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=n_o}^{\infty} a_n$ is divergent.

Remark If $\lim_{n\to\infty} a_n = 0$ the n^{th} Term Divergence Test ***FAILS*** and says nothing about $\sum_{n=n_o}^{\infty} a_n$; it may be ***convergent*** or ***divergent***.

Example Consider the series

$$\sum_{n=1}^{\infty} \frac{n}{n+1}.$$

Since $\lim_{n\to\infty} \frac{n}{n+1} = 1$, this series diverges.

Example The Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

has $\lim_{n\to\infty} \frac{1}{n} = 0$ as the limit, but the series diverges. (We shall show this soon.)

Example The series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^2+n}}$$

diverges since

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = 1 \neq 0$$

Example The series

$$\sum_{n=1}^{\infty} (-1)^n n^2 = 1 - 4 + 9 - 16 + 25 - \cdots$$

diverges because $\lim_{n\to\infty} a_n$ does not exist.

The Integral Test

The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

For the proof hold down the Ctrl key and click on Integral Test

Example The Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$.

Consider $f(x) = \frac{1}{x}$ on $[1,\infty)$; then $f(x)$ is positive and decreasing and

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{\beta\to\infty} \int_1^{\beta} \frac{1}{x} dx = \lim_{\beta\to\infty} \ln x\Big]_1^{\beta} = \lim_{\beta\to\infty} (\ln\beta - \ln 1) \to \infty$$

Since this integral diverges, the given series diverges by the Integral Test.

Example The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$.

Consider $f(x) = \frac{1}{x^p}$ on $[1,\infty)$; then $f(x)$ is positive and decreasing and

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{\beta\to\infty} \int_1^{\beta} \frac{1}{x^p} dx = \lim_{\beta\to\infty} \frac{x^{-p+1}}{-p+1}\Bigg]_1^{\beta} = \lim_{\beta\to\infty} \left(\frac{\beta^{-p+1}}{-p+1} - \frac{1}{-p+1} \right)$$

which diverges if $p < 1$ since in this case $1 - p > 0$ so that we are taking the limit of β to a positive power as that power goes to ∞. The series converges if $p > 1$, since in this case we are taking the limit of β to a negative power as that power goes to ∞. [The case when $p = 1$ reduces the p-series to the Harmonic Series].

Example Show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges.

Solution: Note that the series begins at $n = 2$, since $\ln 1 = 0$. $\frac{1}{n(\ln n)^2} > 0$, so let $f(x) = \frac{1}{x(\ln x)^2}$. Then

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} = \frac{1}{\ln 2}$$

Thus the series converges by the Integral Test.

Integral Test Remainder Estimate

Theorem. The Integral Test Estimate. Suppose that

$$\sum_{n=1}^{\infty} a_n$$

is a series which satisfies the hypotheses of the Integral Test using the function f and which converges to L. Let

$$s_n = a_1 + a_2 + \cdots + a_n$$

be the nth partial sum and let

$$r_n = L - s_n$$

Then

$$\int_{n+1}^{\infty} f(x)dx \le r_n \le \int_n^{\infty} f(x)dx$$

For the proof and examples of this theorem hold down the Ctrl key and click on Integral Test Estimate