# Ma 530

# Infinite Series II

## The Comparison Test

Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $a_n \leq b_n$.

(a) If $\sum b_n$ is convergent; then $\sum a_n$ is also convergent.
(b) If $\sum a_n$ is divergent; then $\sum b_n$ is also divergent.

**Example** Determine whether

$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$$

converges.
Solution: Now $n^2 + 3n > n^2$ so

$$\frac{3}{n^2+3n} < \frac{3}{n^2}.$$

But $\sum \frac{3}{n^2}$ converges, since it is a p-series with $p = 2 > 1$. Hence, the given series converges by comparison.

**Example** Determine whether

$$\sum_{n=3}^{\infty} \frac{1}{n^{\frac{1}{2}} - 2}$$

converges.

Since $n^{\frac{1}{2}} - 2 < n^{\frac{1}{2}}$ for $n \geq 3,$ then $\frac{1}{n^{\frac{1}{2}} - 2} > \frac{1}{n^{\frac{1}{2}}}$. However, $\sum \frac{1}{n^{\frac{1}{2}}}$ diverges (why? ).
Hence the original series diverges.

**Example** Show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$$

converges.

$$\frac{1}{n(n+1)(n+2)} < \frac{1}{n^3}$$

$\sum \frac{1}{n^3}$ is a $p$ –series with $p = 3,$ and hence converges. This implies by the Comparison Test that the original series converges.

**Example** Test the series

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

for convergence.

$$\begin{aligned} n! &= n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 \\ &\geq 2 \cdot 2 \cdot 2 \cdots 2 \cdot 2 \cdot 1 \\ &= 2^{n-1} \end{aligned}$$

Thus

$$\frac{1}{n!} \leq \frac{1}{2^{n-1}} \quad \text{for } n \geq 1$$

Hence

$$\sum_{n=0}^{\infty} \frac{1}{n!} \leq 1 + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \sum_{n=0}^{\infty} \frac{1}{2^n}$$

But this last series is a convergent geometric series. Since the original series and this series are positive, the original series converges by the Comparison Test.

For more on the Comparison Test hold down the Ctrl key and click on Comparison Test

# The Alternating Series Test

Definition A series $\sum_{n=n_o}^{\infty} a_n$ is alternating if $a_n = (-1)^n b_n$ (or $a_n = (-1)^{n+1} b_n$) where $b_n$ is positive for all $n$. Notice that $b_n = |a_n|$, so $b_n$ is positive..

The Alternating Series Test: Given the series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, if

$$(i) b_{n+1} \leq b_n \text{ for all } n$$

and

$$(ii) \lim_{n\to\infty} b_n = 0;$$

then the series is convergent.

The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

has $b_n = \frac{1}{n}$ and therefore satisfies (i) $b_{n+1} < b_n$ because

$$\frac{1}{n+1} < \frac{1}{n}$$

and (ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

Hence, the Alternating Harmonic Series converges.

**Example** The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n-1} = 1 - \frac{2}{3} + \frac{3}{5} - \frac{4}{7} + \frac{5}{9} - \cdots$$

is an alternating series. Let $b_n = \frac{n}{2n-1}$. Then

$$b_n = \frac{n}{2n-1} > \frac{n+1}{2n+1} = b_{n+1}$$

for $n \geq 1$. But

$$\lim_{n\to\infty} b_2 = \frac{1}{2} \neq 0$$

so the Alternating Series Test does not apply. We see that the series diverges because the $n$th term does not go to zero.

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Alternating Series Estimation Series: If $s = \sum(-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

$$(i) 0 < b_{n+1} \leq b_n \text{ and } (ii) \lim_{n\to\infty} b_n = 0;$$

then

$$|R_n| = |S - S_n| \leq b_{n+1}$$

where $R_n$ is the remainder if we use the partial sum $S_n$ as an approximation to the total sum $S$.
For more on alternating series hold down the Ctrl key and click on Alternating Series.

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# Absolute Convergence

Definition Given a series $\sum_{n=n_o}^{\infty} a_n$, its related absolute series is $\sum_{n=n_o}^{\infty} |a_n|$.

Proposition ***The Absolute Convergence Test***

If the related absolute series $\sum_{n=n_o}^{\infty} |a_n|$ is convergent, then the original series $\sum_{n=n_o}^{\infty} a_n$ is convergent.

Remark The Absolute Convergence Test says nothing about $\sum_{n=n_o}^{\infty} a_n$ if the related absolute series $\sum_{n=n_o}^{\infty} |a_n|$ is divergent.

Remark Note that $\sum |a_n| = |a_1| + |a_2| + \cdots$. If a series is absolutely convergent, then the proposition tells us that it is convergent. The converse is not true, namely $\sum a_n$ can converge, but $\sum |a_n|$ may diverge. Can you give an example of this case?

Definition If $\sum_{n=n_o}^{\infty} |a_n|$ is convergent, then $\sum_{n=n_o}^{\infty} a_n$ is called absolutely convergent. ( So the proposition says “An absolutely convergent series is convergent.”)

If $\sum_{n=n_o}^{\infty}|a_n|$ is divergent but $\sum_{n=n_o}^{\infty} a_n$ is convergent, then $\sum_{n=n_o}^{\infty} a_n$ is called conditionally convergent.

If $\sum_{n=n_o}^{\infty}|a_n|$ is divergent and $\sum_{n=n_o}^{\infty} a_n$ is divergent, then $\sum_{n=n_o}^{\infty} a_n$ is just called divergent.

**Example** Thus

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

is conditionally convergent.

**Example** Test the series

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

for convergence.

Note that series has both positive and negative terms but it is not alternating. We shall look at the series of absolute values

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|$$

Now

$$\left| \frac{\sin n}{n^2} \right| \leq \frac{1}{n^2}$$

But $\sum \frac{1}{n^2}$ converges. (Why?). So $\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|$ is convergent by the Comparision Test. Since absolute convergence implies convergence, we have the $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is convergent.

For more on absolute Convergence hold down the Ctrl key and click on Absolute Convergence.

## The Ratio Test

Recall

**Definition:** A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

Ratio Test: Given a series $\sum a_n$, let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

(a) If $L < 1$, then the series converges absolutely.
(b) If $L > 1$, then the series is divergent.
(c) If $L = 1$, then the test fails (i.e., is inconclusive.)

Remark For an example of (c) consider $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. Both have a value of 1 for $L$. Yet, the former diverges, whereas the latter converges.

**Example** Determine whether

$$\sum_{n=1}^{\infty} \frac{(n+1)5^n}{n3^{2n}}$$

is convergent.

Solution: We apply the Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+2)5^{n+1}}{(n+1)3^{2n+2}}}{\frac{(n+1)5^n}{n3^{2n}}} \right| = \lim_{n\to\infty} \left| \frac{(n+2)5^{n+1}}{(n+1)3^{2n+2}} \cdot \frac{n3^{2n}}{(n+1)5^n} \right| = \lim_{n\to\infty} \frac{5}{3^2} \left( \frac{n^2+2n}{n^2+2n+1} \right) = \frac{5}{9} < 1.$$

Therefore, the given series converges by the Ratio Test.

**Example** Test the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{n!}$$

for convergence or divergence.

Solution: Using the Ratio Test we have

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}4^{n+1}}{(n+1)!}}{\frac{(-1)^n 4^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{4^{n+1}}{(n+1)!} \cdot \frac{n!}{4^n} \right| = \lim_{n\to\infty} \frac{4}{n+1} = 0$$

Thus this series converges absolutely, since $L < 1$.

**Example** Test the series

$$\sum_{n=1}^{\infty} \frac{7^n}{n^2}$$

for convergence or divergence.

Solution: Again the Ratio Test may be applied. We have

$$L = \lim_{n\to\infty} \left| \frac{\frac{7^{n+1}}{(n+1)^2}}{\frac{7^n}{n^2}} \right| = \lim_{n\to\infty} \left| \frac{7^{n+1}}{(n+1)^2} \cdot \frac{n^2}{7^n} \right| = \lim_{n\to\infty} 7 \cdot \left( \frac{n^2}{n^2+2n+1} \right) = 7$$

Since $L > 1$, this series diverges.