Ma 530

Laplace Transforms

Review of Improper Integrals.

Since the Laplace Transform is an improper integral we recall:

Definition. If $f(x)$ is defined on $a \le x < \infty$, then we define $\int_a^\infty f(x)dx$ by

$$\int_a^\infty f(x)dx = \lim_{R\to\infty} \int_a^R f(x)dx$$

provided this limit exists. If the limit exists, we say the improper integral converges, otherwise it diverges.

Example 1) $\int_0^\infty x^n dx$ diverges for $n > -1$, since

$$\int_0^\infty x^n dx = \lim_{R\to\infty} \frac{x^{n+1}}{n+1}\Big|_0^R \to \infty$$

2) $\int_1^\infty x^{-n} dx$ converges for $n > 1$, since

$$\int_1^\infty x^{-n} dx = \lim_{R\to\infty} \int_1^R \frac{1}{x^n} dx = \lim_{R\to\infty} \frac{x^{-n+1}}{1-n}\Big|_1^R$$
$$= \lim_{R\to\infty} \left\{ \frac{R^{-n+1}}{1-n} - \frac{1}{1-n} \right\} = +\frac{1}{n-1}$$

.

The Comparison Test for Convergence.

If $0 \le f(x) \le g(x)$ on $a \le x < \infty$ and $\int_a^\infty g(x)dx$ converges $\Rightarrow \int_a^\infty f(x)dx$ converges.

Laplace Transforms

Definition. Let f be defined on $0 \le x < \infty$. Then the Laplace Transform of $f(x)$, denoted by $\mathrm{L}\{f(x)\}$ or $\hat{f}(s)$ is defined by

$$\mathrm{L}\{f(x)\} = \hat{f}(s) = \int_0^\infty e^{-sx} f(x)dx$$

provided that the integral converges for at least one value of s.

Remarks. (1) $f(x)$ is transformed to a new function $\hat{f}(s)$

(2) It can be shown that if $\hat{f}(s)$ exists for $s = s_0$ it exists $\forall\ s > s_0$.

Example $\mathrm{L}\{x\}$

$$\mathrm{L}\{x\} = \int_0^\infty xe^{-sx}dx$$

To evaluate this integral we use integration by parts. Let
$u = x \quad dv = e^{-sx} \Rightarrow v = -\frac{1}{s}e^{-sx}$. Then

$$\begin{aligned}\int_0^\infty xe^{-sx}dx &= \lim_{R\to\infty} x(-\tfrac{1}{s})e^{-sx}\Big|_0^R - \int_0^\infty -\tfrac{1}{s}e^{-sx}dx \\ &= 0 + \frac{1}{s}\int_0^\infty e^{-sx}dx = \lim_{R\to\infty} \frac{1}{s}\frac{e^{-sx}}{-s}\Big|_0^R = -\frac{1}{s^2}[0-1] = \frac{1}{s^2}\end{aligned}$$

Thus

$$\mathrm{L}\{x\} = \frac{1}{s^2}$$

.

Example

$$\mathrm{L}\{e^{ax}\} = \frac{1}{s-a}$$

Remark.

$$\begin{aligned}\mathrm{L}\{af(x)+bg(x)\} &= \int_0^\infty \{af(x)+bg(x)\}e^{-sx}dx \\ &= a\int_0^\infty f(x)e^{-sx}dx + b\int_0^\infty g(x)e^{-sx}dx \\ &= a\mathrm{L}\{f(x)\} + b\mathrm{L}\{g(x)\}\end{aligned}$$

Example $\mathrm{L}\{\cos bx\} = ?$

Consider $e^{ibx} = \cos bx + i\sin bx$.

$$\mathrm{L}\{e^{ibx}\} = \mathrm{L}\{\cos bx\} + i\mathrm{L}\{\sin bx\}$$

Therefore we can find $\mathrm{L}\{e^{ibx}\}$, and we take $\mathrm{Re}\,\mathrm{L}\{e^{ibx}\} \Rightarrow$ we will get $\mathrm{L}\{\cos bx\}$.

$$\mathrm{L}\{e^{ibx}\} = \frac{1}{s-ib}$$

Now the real part of $\mathrm{L}\{e^{ibx}\} = \mathrm{L}\{\cos bx\}$. Thus

$$\frac{1}{s-ib} \times \frac{s+ib}{s+ib} = \frac{s+ib}{s^2+b^2} = \frac{s}{s^2+b^2} + i\frac{b}{s^2+b^2}$$

$\Rightarrow$

$$\mathrm{L}\{\cos bx\} = \frac{s}{s^2+b^2}$$

Also

$$\mathrm{L}\{\sin bx\} = \frac{b}{s^2+b^2}.$$

Use of the Laplace Transform to solve differential equations.

Example

$$y' - 2y = e^{-3x} \quad y(0) = 1$$

We first solve this initial value problem by techniques taught earlier.

The integrating factor for this equation is $e^{-\int 2dx} = e^{-2x}$. Multiplying by this

$\Rightarrow \frac{d}{dx}(ye^{-2x}) = e^{-5x} \Rightarrow ye^{-2x} = -\frac{1}{5}e^{-5x} + c \Rightarrow y = -\frac{1}{5}e^{-3x} + ce^{2x}$

$y(0) = 1 \Rightarrow -\frac{1}{5} + c = 1 \quad \Rightarrow \quad c = \frac{6}{5} \quad \Rightarrow$

$$y = -\frac{1}{5}e^{-3x} + \frac{6}{5}e^{2x}.$$

Solution by Laplace Transforms.

Take L of both sides. Since $L\{e^{ax}\} = \frac{1}{s-a}$

$\Rightarrow$

$$L\{y' - 2y\} = L\{e^{-3x}\} = \frac{1}{s+3}$$

or

$$L\{y'\} - 2L\{y\} = \frac{1}{s+3}$$

Later we shall show that

$$L\{y'\} = sL\{y\} - y(0).$$

$\Rightarrow$

$$sL\{y\} - 1 - 2L\{y\} = \frac{1}{s+3}$$

$\Rightarrow$

$$L\{y\}(s-2) = \frac{1}{s+3} + 1$$

$\Rightarrow$

$$L\{y\} = \frac{1}{s-2} - \frac{1}{(s-2)(s+3)}$$

Now using partial fractions we have

$$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$$

$\Rightarrow A = \frac{1}{5}$ and $B = -\frac{1}{5}$.

$\Rightarrow$

$$L\{y\} = \frac{1}{s-2} + \frac{\frac{1}{5}}{s-2} + \frac{-\frac{1}{5}}{s+3} = \frac{\frac{6}{5}}{s-2} + \frac{-\frac{1}{5}}{s+3}$$

Since $L\{e^{2x}\} = \frac{1}{s-2}$ and $L\{e^{-3x}\} = \frac{1}{s+3}$

$\Rightarrow$

$$\begin{aligned} L\{y\} &= L\{\frac{6}{5}e^{2x}\} + L\{-\frac{1}{5}e^{-3x}\} \\ &= L\{\frac{6}{5}e^{2x} - \frac{1}{5}e^{-3x}\} \end{aligned}$$

$\Rightarrow$

$$y = -\frac{1}{5}e^{-3x} + \frac{6}{5}e^{2x}$$

as before.

Remark. If $L\{f(x) = F(s)$, we may write $L^{-1}\{F(s)\} = f(x)$.

Question: Is there more than one such $f(x)$ for a given $F(s)$? Answer–No.

Theorem. If $L\{f(x)\} \equiv L\{g(x)\}$ and f and g are continuous for $0 \le x < \infty$, then $f(x) = g(x)$.
L^{-1} is called inverse Laplace Transform.

Definition. $f(x)$ is of exponential order α if $f(x)$ is continuous for $0 \le x < \infty$ and

$$|f(x)| < ce^{\alpha x} \; 0 \le x < \infty$$

where c and α are constants.

Note: The above definition says that a function of exponential order α cannot go to infinity "faster" than $e^{\alpha x}$. We shall denote the set of all functions of exponential order α by E_α .

Theorem. If $\int_a^\infty |f(x)|dx$ converges, then $\int_a^\infty f(x)dx$ converges.
Theorem. If $f(x) \in E_\alpha$ then $L\{f(x)\}$ exists for $s > \alpha$.
Proof. $L\{f(x)\} = \int_0^\infty e^{-sx}f(x)dx \Rightarrow$

$$\begin{aligned}\int_0^\infty e^{-sx}f(x)dx &\le \int_0^\infty e^{-sx} \; | \; f(x) \; | \; dx \\ &\le c \int_0^\infty e^{-sx}e^{\alpha x}dx \\ &\le c\int_0^\infty e^{-(s-\alpha)x}dx = c\,\frac{-e^{-(s-\alpha)x}}{s-\alpha}\Big|_0^\infty = c\frac{1}{s-\alpha} \quad \text{for } s > \alpha\end{aligned}$$

therefore $\int_0^\infty |e^{-sx}f(x)|dx$ converges $\Rightarrow \int_0^\infty e^{-sx}f(x)dx$ also converges. $\Rightarrow L\{f(x)\}$ exists.

Example $\sin x \in E_1$

$$|\sin x| \le ce^x \; c \ge 1.$$

In our solution of the first order differential equation in the example above we used the fact that

$$L\{y'(x)\} = sL\{y(x)\} - y(0).$$

We now prove this.

Theorem. If $f(x) \in E_\alpha$ and $f'(x)$ is continuous, then $L\{f'(x)\}$ exists for $s > \alpha$ and
$L\{f'(x)\} = sL\{f(x)\} - f(0)$.
Proof. $L\{f'(x)\} = \int_0^\infty e^{-sx}f'(x)dx$.
Integrate by parts $\Rightarrow u = e^{-sx} \;\; dv = f'(x)dx \;\; v = f(x)$.
$L\{f'(x)\} = e^{-sx}f(x) \; \Big|_0^\infty + s\int_0^\infty f(x)e^{-sx}dx = \lim_{R\to\infty} \left[e^{-sx}f(x) \; \Big|_0^R \right] + s\int_0^\infty f(x)e^{-sx}dx = \lim_{R\to\infty} [e^{-sR}f(R)] -$
$f(0) + sL\{f(x)\}$.
Claim $\lim_{R\to\infty} e^{-sR}f(R) = 0$ since $f \in E_\alpha$ and $s > \alpha$. Now

$$e^{-sR} \; |f(R)| \le ce^{-sR}e^{\alpha R} = ce^{-(s-\alpha)R} \to 0$$

since $s - \alpha > 0$. Therefore
$\mathrm{L}\{f'(x)\} = s\mathrm{L}\{f(x)\} - f(0)$ and $\mathrm{L}\{f'\}$ exists.

Remark. If $f''(x)$ is continuous and $f'(x) \in \mathrm{E}_\alpha$ we may apply the above theorem to f' and $f'' \Rightarrow$

$$\mathrm{L}\{f''(x)\} = s\mathrm{L}\{f'(x)\} - f'(0) = s[s\mathrm{L}\{f(x)\} - f(0)] - f'(0)$$

Hence

$$\mathrm{L}\{f''\} = s^2\mathrm{L}\{f(x)\} - sf(0) - f'(0).$$

In general

$$\mathrm{L}\{f^{(n)}(x)\} = s^n\widehat{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0)\ldots f^{(n-1)}(0).$$

We want to use the Laplace Transform to solve D.E.'s. To do this we need the Transform of many functions. Therefore we shall develop some results which allow us to find more transforms.
So far we have

$$\mathrm{L}\{e^{ax}\} = \frac{1}{s-a}$$

$$\mathrm{L}\{1\} = \frac{1}{s}$$

$$\mathrm{L}\{x\} = \frac{1}{s^2}$$

$$\mathrm{L}\{\cos bx\} = \frac{s}{s^2+b^2}$$

$$\mathrm{L}\{\sin bx\} = \frac{b}{s^2+b^2}.$$

Theorem. If $f(x) \in \mathrm{E}_\alpha$ and

$$F(x) = \int_0^x f(t)dt,$$

then $F(x) \in \mathrm{E}_\alpha$ and

$$\mathrm{L}\left\{\int_0^x f(t)dt\right\} = \mathrm{L}\{F(x)\} = \tfrac{1}{s}\widehat{f}(s) = \tfrac{1}{s}\mathrm{L}\{f(x)\}.$$

Remark. Assume theorem is true. Then

$$\mathrm{L}\{x^2\} = \mathrm{L}\left\{2\int_0^x tdt\right\} = 2\mathrm{L}\left\{\int_0^x tdt\right\} = \tfrac{2}{s}\mathrm{L}\{x\} = \frac{2}{s^3}.$$

Note that here $F = x^2$ and $f = 2t$.
Proof of Theorem. We shall first show $F(x) \in \mathrm{E}_\alpha$.

$$|F(x)| \le \int_0^x |f(t)|dt \le c\int_0^x e^{\alpha t}dt = \tfrac{c}{\alpha}[e^{\alpha x} - 1] \le \tfrac{c}{\alpha}e^{\alpha x} = Ce^{\alpha x}.$$

where $\frac{c}{\alpha} = C$.
Also $\mathrm{L}\{F'(x)\} = s\mathrm{L}\{F(x)\} - F(0) \Rightarrow$

$$\mathrm{L}\{f(x)\} = s\mathrm{L}\{F(x)\} - F(0) = s\mathrm{L}\{F(x)\}$$

since $F(0) = 0$.

$\Rightarrow$

$$\mathrm{L}\{F(x)\} = \mathrm{L}\left\{\int_0^x f(t)dt\right\} = \frac{1}{s}\widehat{f}(s).$$

Theorem. If $f \in \mathrm{E}_\alpha$ then

$$\mathrm{L}\{e^{-ax}f(x)\} = \widehat{f}(s+a)$$

where $s > \alpha - a$.

Proof. $\mathrm{L}\{e^{-ax}f(x)\} = \int_0^\infty e^{-sx}e^{-ax}f(x)dx = \int_0^\infty e^{-(s+a)x}f(x)dx = \widehat{f}(s+a)$.

Example $\mathrm{L}\{e^{-ax}x^2\} = ?$

$$\mathrm{L}\{x^2\} = \frac{2}{s^3}$$

$\Rightarrow$

$$\mathrm{L}\{e^{-ax}x^2\} = \frac{2}{(s+a)^3}.$$

Remark. Consider

$$\frac{d}{ds}\widehat{f}(s) = \frac{d}{ds}\int_0^\infty e^{-sx}f(x)dx = \int_0^\infty e^{-sx}[-xf(x)]dx = \mathrm{L}\{-xf(x)\}.$$

The above is O.K. if $f \in \mathrm{E}_\alpha$. It can be used to get many Laplace transforms.

Example $\mathrm{L}\{x\cos bx\}$.

$\mathrm{L}\{\cos bx\} = \frac{s}{s^2+b^2}$ so $f(x) = \cos bx$ and $\widehat{f}(s) = \frac{s}{s^2+b^2}$

Thus

$$\mathrm{L}\{x\cos bx\} = -\frac{d}{ds}\left[\frac{s}{s^2+b^2}\right] = -\frac{1}{s^2+b^2} + s(s^2+b^2)^{-2}(2s)$$

$$= -\frac{1}{s^2+b^2} + \frac{2s^2}{(s^2+b^2)^2} = \frac{-s^2-b^2+2s^2}{(s^2+6^2)^2}$$

Hence

$$\mathrm{L}\{x\cos bx\} = \frac{s^2-b^2}{(s^2+b^2)^2}.$$

Remark. In general if $f \in \mathrm{E}_\alpha$, then

$$\mathrm{L}\{(-x)^n f(x)\} = \widehat{f}^{(n)}(s).$$

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Solutions of linear equations with constant coefficients

Consider the Initial Value Problem

$$ay'' + by' + cy = g(x)$$

$$y(0) = \alpha \quad y'(0) = \beta$$

where a, b, and c are constants. It may be shown that y, y' and y'' are of exponential order, and hence their transforms exist. Taking L of both sides of the differential equation

$\Rightarrow$

$$a\mathrm{L}\{y''\} + b\mathrm{L}\{y'\} + c\mathrm{L}\{y\} = \mathrm{L}\{f(x)\}$$

$$a[s^2\mathrm{L}\{y\} - sy(0) - y'(0)] + b[s\mathrm{L}\{y\} - y(0)] + c\mathrm{L}\{y\} = \mathrm{L}\{g(x)\}$$

$$as^2\hat{y}(s) - sa\alpha - a\beta + bs\hat{y} - b\alpha + c\hat{y}(s) = \hat{g}(s)$$

$$(as^2 + bs + c)\hat{y}(s) - sa\alpha - b\alpha - a\beta = \hat{g}(s)$$

Thus

$$\hat{y}(s) = \frac{\hat{g}(s) + \alpha(as + b) + a\beta}{as^2 + bs + c}$$

Therefore once we find L^{-1} of the right hand side we will have $y(x)$.

Example

$$y'' - y' - 2y = 0 \qquad y(0) = 1\ y'(0) = 0$$

$\Rightarrow$

$$\mathrm{L}\{y''\} - \mathrm{L}\{y'\} - 2\mathrm{L}\{y\} = 0$$

$\Rightarrow$

$$s^2\mathrm{L}\{y\} - sy(0) - y'(0) - [s\mathrm{L}\{y\} - y(0)] - 2\mathrm{L}\{y\} = 0$$

$\Rightarrow$

$$s^2\hat{y}(s) - s - s\hat{y}(s) + 1 - 2\hat{y}(s) = 0$$

$$(s^2 - s - 2)\ \hat{y}(s) = s - 1$$

$\Rightarrow$

$$\hat{y}(s) = \frac{s-1}{s^2 - s - 2}$$

$$\frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

$\Rightarrow$

$$A = \frac{1}{3} \text{ and } B = \frac{2}{3}$$

Thus

$$\hat{y}(s) = \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1}$$

and hence

$$y(x) = \frac{1}{3}e^{2x} + \frac{2}{3}e^{-x}.$$

Example Solve

$$y'' + y = \sin 2t \qquad y(0) = 0 \qquad y'(0) = 1$$

Solution:

Recall that $\mathrm{L}\{\sin at\} = \dfrac{a}{s^2 + a^2}$. Taking transforms of the equation $\Rightarrow$

$$s^2 \hat{y}(s) - sy(0) - y'(0) + \hat{y}(s) = \mathrm{L}\{\sin 2t\}.$$

Thus

$$(s^2 + 1)\hat{y}(s) = s \bullet 0 + 1 + \frac{2}{s^2 + 4}$$

$$\hat{y}(s) = \frac{1}{s^2 + 1} + \frac{2}{(s^2 + 1)\ (s^2 + 4)} \qquad (*)$$

To find $\hat{y}(s)$ we must invert $(*)$. Note that

$$\mathrm{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} = \sin t$$

Thus we need to find

$$\mathrm{L}^{-1}\left\{\frac{2}{(s^2 + 1)\ (s^2 + 4)}\right\}$$

We present two approaches.

Approach 1: (using complex variables)

$$\frac{2}{(s^2 + 1)\ (s^2 + 4)} = \frac{2}{(s + i)(s - i)(s + 2i)(s - 2i)} = \frac{A_1}{s + i} + \frac{A_2}{s - i} + \frac{A_3}{s + 2i} + \frac{A_4}{s - 2i} \qquad (*\,*)$$

where $i = \sqrt{-1}$.

To get A_1 we multiply $(*\,*)$ by $(s + i)$ and the set $s = -i$. This yields

$$\frac{2}{-2i(i)(-3i)} = -\frac{1}{3i} = A_1$$

Similarly

$$\frac{2}{2i(3i)(-i)} = \frac{1}{3i} = A_2$$

$$\frac{2}{(-i)(-3i)(-4i)} = \frac{1}{6i} = A_3$$

$$\frac{2}{3i(i)(4i)} = -\frac{1}{6i} = A_4$$

Thus

$$\frac{2}{(s^2+1)\ (s^2+4)} = \frac{-\frac{1}{3i}}{s+i} + \frac{\frac{1}{3i}}{s-i} + \frac{\frac{1}{6i}}{s+2i} + \frac{-\frac{1}{6i}}{s-2i}$$

But

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{-at}$$

so

$$\begin{aligned} L^{-1}\left\{\frac{2}{(s^2+1)\ (s^2+4)}\right\} &= \frac{1}{3i}[-e^{-it}+e^{it}] + \frac{1}{6i}[e^{-2it}-e^{2it}] \\ &= \frac{1}{3i}[-\cos t + i\sin t + \cos t + i\sin t] + \frac{1}{6i}[\cos 2t - i\sin 2t - \cos 2t - i\sin t] \\ &= \frac{2}{3}\sin t - \frac{1}{3}\sin 2t \end{aligned}$$

Thus we have finally that

$$\begin{aligned} y(t) &= L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{\frac{2}{(s^2+1)\ (s^2+4)}\right\} \\ &= \sin t + \frac{2}{3}\sin t - \frac{1}{3}\sin 2t \\ &= \frac{5}{3}\sin t - \frac{1}{3}\sin 2t \end{aligned}$$

Approach 2: (without using complex variables)

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{cs+D}{s^2+4} = \frac{(As+B)(s^2+4)+(Cs+D)(s^2+1)}{(s^2+1)(s^2+4)}$$

$$As^3 + 4As + Bs^2 + 4B + Cs^3 + Ds^2 + Cs + D = 2$$

$A + C = 0 \quad B + D = 0 \quad 4A + C = 0 \quad 4B + D = 2$. Thus $A = -C$ and also $4A = -C \Rightarrow A = C = 0$. In addition, $B = -D$ and hence $-4D + D = 2$. Thus $B = -D = \frac{2}{3}$.

$\Rightarrow$

$$\begin{aligned} y(x) &= \sin t + L^{-1}\left\{\frac{\frac{2}{3}}{s^2+1} + \frac{-\frac{2}{3}}{s^2+4}\right\} \\ &= \sin t + \frac{2}{3}\sin t - \frac{2}{3}L^{-1}\left\{\frac{1}{s^2+4}\right\} \\ &= \frac{5}{3}\sin t - \frac{2}{3}\left(\frac{1}{2}\right)\sin 2t \\ &= \frac{5}{3}\sin t - \frac{1}{3}\sin 2t \end{aligned}$$

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Laplace Transforms and Special Functions

Mathematical models of mechanical or electrical systems often involve functions with discontinuities corresponding to external forces that are turned abruptly on or off. One such simple on-off function is the unit step function at $t = a$; its formula is

$$u(t-a) = \begin{cases} 0 \text{ if } t < a \\ 1 \text{ if } t > a \end{cases}$$

Note: u is often called the Heaviside function.

Theorem. $L\{u(t-a)\} = \frac{e^{-as}}{s}$ for $s > 0,\ a > 0$.

Proof: $L\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)dt = \int_a^\infty e^{-st} \bullet 1dt = \lim_{R\to\infty} \int_a^R e^{-st}dt$

$= \lim_{R\to\infty} \frac{e^{-st}}{s} \mid_a^R = \frac{e^{-as}}{s}$.

Note that $L\{u(t)\} = L\{1\} = \frac{1}{s}$, since $u(t) = 1$ for $t > 0$.

Example Find the Laplace transform of

$$f(t) = \begin{cases} -1 & t < 3 \\ 5 & t > 3 \end{cases}$$

We may write $f(t)$ in terms of $u(t-a)$ as follows:

$$f(t) = -1 + 6u(t-3)$$

Hence

$$L\{f(t)\} = L\{-1\} + 6L\{u(t-3)\} = -\frac{1}{s} + 6\frac{e^{-3s}}{s}.$$

Theorem (Shifting Property). Let $F(s) = L\{f(t)\}$ exist for $s > \alpha \geq 0$. If $a > 0$, then

$$L\{f(t-a)u(t-a)\} = e^{-as}F(s) = e^{-as}L\{f(t)\} \qquad (1)$$

and if $f(t)$ is continuous on $[0,\infty)$, then

$$L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a) \qquad (2)$$

Example Let us put $f(t) = \frac{1}{2}t^2$ in the above theorem. Recalling that $L\{\frac{1}{2}t^2\} = \frac{1}{s^3}$ we have

$$L^{-1}\left\{\frac{e^{-as}}{s^3}\right\} = u(t-a)\frac{1}{2}(t-a)^2 = \begin{cases} 0 & \text{if } t < a \\ \frac{1}{2}(t-a)^2 & \text{if } t > a \end{cases}$$

More often one wants to compute the transform of a function expressed as $g(t)u(t-a)$ rather than $f(t-a)u(t-a)$. To compute $L\{g(t)u(t-a)\}$, identify $g(t)$ with $f(t-a)$, so that $f(t) = g(t+a)$. Equation (1) then becomes

$$L\{g(t)u(t-a)\} = e^{-as}L\{g(t+a)\} \quad (3)$$

Example Find $L\{g(t)\}$ if $g(t) = \begin{cases} 0 & \text{if } t < 3 \\ t^2 & \text{if } t > 3 \end{cases}$

Now $g(t) = t^2u(t-3)$ so that

$$\mathrm{L}\{g(t)\} = e^{-3s}\mathrm{L}\{(t+3)^2\} = e^{-3s}\mathrm{L}\{t^2+6t+9\} = e^{-3s}\left(\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right)$$

Example Find $\mathrm{L}\{f(t)\}$ if

$$f(t) = \begin{cases} \cos 2t & \text{if } 0 \le t < 2\pi \\ 0 & \text{if } t > 2\pi \end{cases}$$

Note that

$$f(t) = [1 - u(t-2\pi)]\cos 2t = \cos 2t - u(t-2\pi)\cos 2t.$$

Hence

$$\begin{aligned} \mathrm{L}\{f(t)\} &= \mathrm{L}\{\cos 2t\} - \mathrm{L}\{u(t-2\pi)\cos 2t\} \\ &= \frac{s}{s^2+4} - e^{-2\pi s}\mathrm{L}\{\cos(2t+2\pi)\} \\ &= \frac{s}{s^2+4} - e^{-2\pi s}\mathrm{L}\{cos(2t)\} \quad \text{since cosine has period } 2\pi. \\ &= \frac{s(1-e^{-2\pi s})}{s^2+4} \end{aligned}$$

Example A mass that weighs 32 lb. (mass $m = 1$ slug) is attached to the free end of a long light spring that is stretched 1 ft. by a force of 4 lb. ($k = 4$ lb./ft.). The mass is initially at rest in its equilibrium position. Beginning at time $t = 0$ (seconds), an external force $F(t) = \cos 2t$ is applied to the mass, but at time $t = 2\pi$ this force is turned off (abruptly discontinued), and the mass is allowed to continue its motion unimpeded. Find the resulting position function $x(t)$ of the mass.

We need to solve the initial value problem

$$x'' + 4x = f(t); \quad x(0) = x'(0) = 0$$

where $f(t)$ if given by the function in the previous example. Taking Laplace transforms of the differential equation leads to

$$(s^2+4)\mathrm{L}\{x(t)\} = \frac{s(1-e^{-2\pi s})}{s^2+4}$$

so that

$$\mathrm{L}\{x(t)\} = \frac{s}{(s^2+4)^2} - e^{-2\pi s}\frac{s}{(s^2+4)^2}$$

Now since $\mathrm{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{4}t\sin 2t$. (Recall the formula $\mathrm{L}\{(-t)^n f(t)\} = \widehat{f}^{(n)}(s)$, and use it with $n = 1$ and $f(t) = \sin 2t$. It then follows from (2) above that

$$\begin{aligned} x(t) &= \frac{1}{4}t\sin 2t - u(t-2\pi)\cdot\frac{1}{4}(t-2\pi)\sin 2(t-2\pi) \\ &= \frac{1}{4}[t - u(t-2\pi)\cdot(t-2\pi)]\sin 2t \end{aligned}$$

This last expression may be written as

$$x(t) = \begin{cases} \frac{1}{4}t\sin 2t & \text{if } t < 2\pi \\ \frac{\pi}{2}\sin 2t & \text{if } t \ge 2\pi \end{cases}$$

We see that the mass oscillates with circular frequency $\omega = 2$ and with linearly increasing amplitude until the force is removed at time $t = 2\pi$. Thereafter, the mass continues to oscillate with the same frequency but with constant amplitude $\frac{\pi}{2}$. The force $f(t) = \cos 2t$ would produce pure resonance if continued indefinitely, but we see its effect ceases immediately when it is turned off.

Transforms of Periodic Functions

Periodic forcing terms in mechanical and electrical systems are often more complicated than pure sines or cosines. Hence we have

Definition. A function $f(t)$ is to be periodic of period T if

$$f(t+T) = f(t)$$

$\forall\ t$ in the domain of f.

Example. $f(t) = \begin{cases} -1 & 0 < t < 2 \\ 0 & 2 < t < 4 \end{cases}$ where the period of $f(t)$ is 4.

Theorem. If $f(t)$ has period T and is piecewise continuous on $[0, T]$, then

$$\mathrm{L}\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\,dt \quad (4)$$

Example Find the Laplace Transform of the periodic function given in the example above. Here $T = 4$, so that

$$\begin{aligned} \mathrm{L}\{f(t)\} &= \frac{1}{1-e^{-4s}} \int_0^4 e^{-st} f(t)\,dt = \frac{1}{1-e^{-4s}} \int_0^2 e^{-st}(-1)\,dt \\ &= \frac{1}{1-e^{-4s}} \left(\frac{1}{s}\right) [e^{-2s} - 1] = -\left(\frac{1}{s}\right)\left(\frac{1}{1+e^{-2s}}\right) \end{aligned}$$

Product of Transform Functions: Convolution

It is often desirable to have a function $h(x)$ such that if $f(x)$ and $g(x)$ are two functions having Laplace transforms $\widehat{f}$ and $\widehat{g}$, then

$$\mathrm{L}\{h(x)\} = \mathrm{L}\{f(x)\}\ \mathrm{L}\{g(x)\}$$

Example: $\mathrm{L}\{x\} = \frac{1}{s^2}$ and $\mathrm{L}\{1\} = \frac{1}{s}$

Hence $\mathrm{L}\{1\}\mathrm{L}\{x\} = \frac{1}{s} \cdot \frac{1}{s^2} = \frac{1}{s^3} \neq \mathrm{L}\{1 \cdot x\} = \mathrm{L}\{x\}$. Thus we see that $h \neq fg$.

The following then gives the expression for $h(x)$.

Theorem. If $f(x) \in E_\alpha$ and $g(x) \in E_\alpha$, then the function

$$h(x) = \int_0^x f(x-t)g(t)\,dt \equiv f(x) * g(x)$$

exists and

$$\mathrm{L}\{h(x)\} = \mathrm{L}\{f(x) * g(x)\} = \widehat{f}(s)\widehat{g}(s)$$

Remarks. $f * g$ is called the convolution of f and g. It may be shown that

$f * g = g * f$

$(f * g) * h = f * (g * h)$

$f * (g + h) = f * g + f * h$

$(cf) * g = f * (cg) = c(f * g)$

To see $f * g = g * f$ note that

$\mathrm{L}\{g * f\} = \mathrm{L}\{g\}\mathrm{L}\{f\} = \mathrm{L}\{f\}\mathrm{L}\{g\} = \mathrm{L}\{f * g\}$

Since $L\{h_1\} = L\{h_2\} \Leftrightarrow h_1 = h_2$ for functions h_1 and h_2 which are continuous, $\Rightarrow g * f = f * g$.

Example $L\{1\} = \frac{1}{s} \quad \Rightarrow L\{1\}L\{1\} = \frac{1}{s^2}$
$1 * 1 = \int_0^x 1 \cdot dx = x$ and $L\{x\} = \frac{1}{s^2}$
Example Find

$$L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$$

Since

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos ax$$

and

$$\frac{s^2}{(s^2+4)^2} = \left(\frac{s}{s^2+4}\right)\left(\frac{s}{s^2+4}\right)$$

, therefore we want $h(x)$ such that

$$L\{h\} = \frac{s^2}{(s^2+4)^2} = \frac{s}{s^2+4)} \times \frac{s}{s^2+4}$$

$L\{h\} = L\{\cos 2x\}\, L\{\cos 2x\} \Rightarrow$

$$\begin{aligned}
h(x) &= \cos 2x * \cos 2x = \int_0^x \cos 2(x-t)\cos 2t dt \\
&= \int_0^x (\cos 2x \cos 2t + \sin 2x \sin 2t)\ \cos 2t dt \\
&= \cos 2x \int_0^x \cos^2 2t dt + \sin 2x \int_0^x \sin 2t \cos 2t dt \\
&= \cos 2x \int_0^x \left(\frac{1+\cos 4t}{2}\right) dt + \frac{1}{2}\sin 2x \sin^2 2t\Big|_0^x \\
&= \frac{\cos 2x}{2}\left[t + \frac{\sin 4t}{4}\right]_0^x + \frac{1}{4}\sin^3 2x \\
&= \frac{x}{2}\cos 2x + \frac{\cos 2x \sin 4x}{8} + \frac{\sin^3 2x}{4} \\
&= \frac{x}{2}\cos 2x + \frac{\cos^2 2x \sin 2x}{4} + \frac{\sin^3 2x}{4} = \frac{x}{2}\cos 2x + \frac{\sin 2x}{4}
\end{aligned}$$

Example Solve the integral equation

$$f(x) = 1 + \int_0^x f(t)\sin(x-t)dt$$

The equation may be rewritten as

$$f(x) = 1 + f * \sin x$$

Taking the Laplace of this equation yields

$$\hat{f}(s) = \frac{1}{s} + \hat{f} \times \frac{1}{s^2+1}$$

and hence

$$\hat{f}\left[1 - \frac{1}{s^2+1}\right] = \frac{1}{s}$$

$\Rightarrow$

$$\hat{f} = \frac{s^2+1}{s^3} = \frac{1}{s} + \frac{1}{s^3}$$

$\Rightarrow$

$$f = 1 + \frac{x^2}{2},$$

since $\mathrm{L}\left\{\frac{x^n}{n!}\right\} = \frac{1}{s^{n+1}}$.

Laplace Transforms for Partial Differential Equations

Consider the partial differential equation

$$\frac{\partial}{\partial x}\left(k(x)\frac{\partial\phi}{\partial x}\right) - q(x)\phi = r(x)\left\{\alpha\frac{\partial^2\phi}{\partial t^2} + \beta\frac{\partial\phi}{\partial t}\right\} \tag{1}$$

for $\phi(x,t)$, where $k(x), r(x)$, and $q(x)$ are continuously differentiable functions and α and β are constants. We assume that there is an interval (a,b) where $k(x) \neq 0$ and $r(x) \neq 0$. Written out (1) is

$$k\phi_{xx} + k'\phi_x - q\phi = r\alpha\phi_{tt} + r\beta\phi_t$$

Together with (1) we are given the following initial and boundary conditions:

$$\begin{aligned} \phi(x,0) &= f(x) \quad && a < x < b \\ \phi_t(x,0) &= v(x) \quad && a < x < b \\ \phi(a,t) &= g(t) \quad && t > 0 \\ \phi(b,t) &= h(t) \quad && t > 0 \end{aligned}$$

To solve this problem we define the Laplace transform of $\phi(x,t)$ as

$$\mathrm{L}\{\phi(x,t)\} = \overline{\phi}(x,s) = \int_0^\infty \phi(x,t)e^{-st}dt$$

Thus $\overline{\phi}$ is a function of x and s. Let $\overline{\phi}'$ denote differentiation of $\overline{\phi}$ with respect to x. Taking the Laplace transform of (1) we have

$$\mathrm{L}\left[\frac{\partial}{\partial x}\left(k(x)\frac{\partial\phi}{\partial x}\right)\right] - \mathrm{L}[q(x)\phi] = \mathrm{L}\left[r(x)\alpha\frac{\partial^2\phi}{\partial t^2}\right] + \mathrm{L}\left[r(x)\beta\frac{\partial\phi}{\partial t}\right] \tag{2}$$

Thus

$$\frac{\partial}{\partial x}\left[k\int_0^\infty \phi_x(x,t)e^{-st}dt\right] - q(x)\overline{\phi} = r\alpha\int_0^\infty \phi_{tt}e^{-st}dt + r\beta\int_0^\infty \phi_t e^{-st}dt$$

From the definition of $\overline{\phi}$ we have

$$\overline{\phi}' = \int_0^\infty \phi_x(x,t)e^{-st}dt$$

Also

$$\begin{aligned} \int_0^\infty \phi_t e^{-st}dt &= \phi e^{-st}\Big|_0^\infty + s\int_0^\infty \phi(x,t)e^{-st}dt \\ &= s\overline{\phi} - \phi(x,0) = s\overline{\phi} - f(x) \end{aligned}$$

and

$$\begin{aligned}\int_0^\infty \phi_{tt}e^{-st}dt &= \phi_t e^{-st}\big|_0^\infty + s\int_0^\infty \phi_t(x,t)e^{-st}dt \\ &= -\phi_t(x,0) + s\overline{\phi} - f(x) \\ &= -v(x) + s^2\overline{\phi} - sf(x)\end{aligned}$$

Hence (2) becomes

$$\begin{aligned}\left(k\overline{\phi}'\right)' - q\overline{\phi} &= r(-\alpha v - \alpha sf + \alpha s^2\overline{\phi} - \beta f + \beta s\overline{\phi}) \\ &= r([\alpha s^2 + \beta s])\overline{\phi} - (\alpha s + \beta)f - \alpha v\end{aligned}$$

Let $\lambda = -\alpha s^2 - \beta s.$ Then we have the ordinary differential equation

$$\left(k\overline{\phi}'\right)' + (\lambda r - q)\overline{\phi} = F(x,s)$$

for $\overline{\phi}$ as a function of $x.$ We also have the following boundary conditions for $\overline{\phi}$:

$$\begin{aligned}\overline{\phi}(a,s) &= \mathrm{L}\{\phi(a,t)\} = \mathrm{L}\{g(t)\} \\ \overline{\phi}(b,s) &= \mathrm{L}\{h(t)\}\end{aligned}$$

Example Solve

$$\phi_{xx} - \phi = 0 \quad \phi(0,t) = 0 \quad \phi(1,t) = t$$

Taking Laplace transforms we have

$$\overline{\phi}'' - \overline{\phi} = 0$$

so

$$\overline{\phi} = Ae^x + Be^{-x}$$

and

$$\overline{\phi}(0,s) = 0 \quad \overline{\phi}(1,s) = \frac{1}{s^2}$$

These conditions give us two equations for A and B :

$$A + B = 0 \quad \text{or} \quad A = -B$$

$$Ae - Ae^{-1} = \frac{1}{s^2}$$

Thus

$$A = \left(\frac{1}{s^2}\right)\left(\frac{1}{e - e^{-1}}\right)$$

so

$$\overline{\phi}(x,s) = \frac{e}{e^2 - 1}\left(\frac{1}{s^2}\right)(e^x - e^{-x})$$

Finally

$$\phi(x,t) = \frac{e}{e^2 - 1}(t)(e^x - e^{-x})$$

Example Consider an infinite string stretched from $x = 0$ to $x = \infty$ which is initially at rest. At time $t = 0$ the $x = 0$ is constrained to move laterally so that its displacement is $g(t)$ for all $t \geq 0.$ If $\phi(x,t)$ represents the displacement of the string at point x at time $t,$ find $\phi(x,t).$

The equation of motion is the wave equation, namely

$$\phi_{xx} = \frac{1}{c^2}\phi_{tt}$$

where $c > 0$ is a constant.
We have

$$\phi(0,t) = g(t) \quad t \geq 0 \quad \text{Boundary condition}$$
$$\phi(x,0) = 0 \quad \phi_t(x,t) = 0 \quad \text{Initial conditions}$$
$$\lim_{x\to\infty}\phi(x,t) = 0 \quad \text{for all } t \geq 0$$

Taking Laplace transforms of the equation we have

$$\begin{aligned}\overline{\phi}'' &= \frac{1}{c^2}\{s^2\overline{\phi} - s\phi(x,0) - \phi_t(x,0)\} \\ &= \frac{s^2}{c^2}\overline{\phi}\end{aligned}$$

or

$$\overline{\phi}'' - \frac{s^2}{c^2}\overline{\phi} = 0$$

Solving this ordinary differential equation we have

$$\overline{\phi}(x,s) = Ae^{\frac{sx}{c}} + Be^{-\frac{sx}{c}}$$

Since $\lim_{x\to\infty}\phi(x,t) = 0 \quad$ for all $t \geq 0$, this implies that $A = 0$ and

$$\overline{\phi}(x,s) = Be^{-\frac{sx}{c}}$$

But $\phi(0,t) = g(t)$ so

$$\overline{\phi}(0,s) = \mathrm{L}\{g(t)\} = \overline{g}(s) = B$$

Therefore

$$\overline{\phi}(x,s) = \overline{g}(s)e^{-\frac{sx}{c}}$$

Finally

$$\phi(x,t) = g\left(t - \frac{x}{c}\right)u\left(t - \frac{x}{c}\right)$$

where u is the Heaviside function.

Note: Recall that $\mathrm{L}^{-1}\left\{e^{-st_0}\bar{f}(s)\right\} = u(t-t_0)f(t-t_0)$.

Therefore

$$\phi(x,t) = \begin{cases} g(t - \frac{x}{c}) \text{ for } x < ct \\ \\ 0 \quad \text{for } x > ct \end{cases}$$

and we see that the disturbance propagates at speed c along the string.