Ma 530 Midterm Exam Solutions 03A

You are to answer **all** of the questions below and have 4 hours to complete the exam. You may use the text and notes while taking this exam.

(1a 30 **pts.)** Solve

$$(x+y)dx + x\ln x dy = 0$$

Solution: We seek an integrating factor u to make the equation exact. Then

$$u(x+y)dx + ux\ln x dy = 0$$

If the equation is to be exact, then the condition $M_y = N_x$ implies

$$u_y(x+y) + u = u_x x\ln x + u\ln x + u$$

Suppose $u_y = 0.$ Then the equation for u is

$$u'x + u = 0$$

or

$$\frac{du}{dx} + \frac{1}{x}u = 0$$

Then a solution is

$$u = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

Multiplying the original DE by this integrating factor gives

$$\left(1 + \frac{y}{x}\right)dx + \ln x dy = 0$$

which is exact. Hence

$$f(x,y) = x + y\ln x + g(y)$$

But

$$f_y = \ln x + g' = \ln x$$

so $g(y) = c,$ where c is a constant. Thus

$$f(x,y) = x + y\ln x + c$$

and the solution is given by

$$x + y\ln x = k$$

where k is a constant.

(1b 30 **pts.)** Solve using Laplace transforms

$$y'' + 2ty' - 4y = 1 \quad y(0) = y'(0) = 0$$

Solution: Taking the Laplace transform of the DE implies

$$s^2Y(s) - sy(0) - y'(0) + 2\mathrm{L}[ty'] - 4Y(s) = \mathrm{L}[1] = \frac{1}{s}$$

Since

$$\mathrm{L}[ty'] = -\frac{d}{ds}\mathrm{L}[y'(t)] = -\frac{d}{ds}[sY(s) - y(0)] = -Y(s) - sY'(s)$$

and the initial conditions are both equal to 0
the equation above becomes

$$s^2Y(s) - 2Y(s) - 2sY'(s) - 4Y(s) = \frac{1}{s}$$

$$(s^2 - 6)Y(s) - 2sY'(s) = \frac{1}{s}$$

Thus we have the following first order DE for $Y(s)$

$$Y'(s) + \left(\frac{3}{s} - \frac{s}{2}\right)Y = -\frac{1}{2s^2}$$

This is a first order linear DE for $Y(s)$ and has the integrating factor

$$e^{\int\left(\frac{3}{s}-\frac{s}{2}\right)ds} = e^{3\ln s - \frac{s^2}{4}} = s^3e^{-\frac{s^2}{4}}$$

Multiplying the DE for Y by this integrating factor yields

$$\frac{d}{ds}\left(s^3e^{-\frac{s^2}{4}}Y\right) = -\frac{s}{2}e^{-\frac{s^2}{4}}$$

so

$$s^3e^{-\frac{s^2}{4}}Y = -\int \frac{s}{2}e^{-\frac{s^2}{4}}ds = e^{-\frac{s^2}{4}} + c$$

$$Y = \frac{1}{s^3} + \frac{c}{s^3}e^{\frac{s^2}{4}}$$

Since $\lim_{s\to\infty} Y(s)$ must be finite, we take $c = 0$, and $Y(s) = \frac{1}{s^3}$. Finally

$$y(t) = \mathrm{L}^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2}$$

$\left(\textbf{2a}\ \ 30\ \textbf{pts.}\right)$

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Solution: Use Variation of Parameters. The characteristic equation is

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$$

Thus the two linearly independent homogeneous solutions are e^{2x} and xe^{2x} and $y_h = c_1e^{2x} + c_2xe^{2x}$. Let

$$y_p = v_1e^{2x} + v_2xe^{2x}$$

$$v_1'e^{2x} + v_2'xe^{2x} = 0$$

$$2v_1'e^{2x} + v_2'(e^{2x} + 2xe^{2x}) = (x+1)e^{2x}$$

Cancelling gives the two equations

$$v_1' + v_2'x = 0$$

$$2v_1' + v_2'(1+x) = (x+1)$$

$$v_1' = \frac{\begin{vmatrix} 0 & x \\ x+1 & 1+2x \end{vmatrix}}{\begin{vmatrix} 1 & x \\ 2 & 1+2x \end{vmatrix}} = -\frac{x^2+x}{1}$$

so

$$v_1 = -\frac{x^3}{3} - \frac{x^2}{2}$$

$$v_2' = \frac{\begin{vmatrix} 1 & 0 \\ 2 & 1+x \end{vmatrix}}{\begin{vmatrix} 1 & x \\ 2 & 1+2x \end{vmatrix}} = 1+x$$

so

$$v_2 = x + \frac{x^2}{2}$$

$$y_p = \left(-\frac{x^3}{3} - \frac{x^2}{2}\right)e^{2x} + \left(x + \frac{x^2}{2}\right)xe^{2x} = \left(\frac{x^3}{6} + \frac{x^2}{2}\right)e^{2x}$$

The general solution is

$$y = c_1e^{2x} + c_2xe^{2x} + \left(\frac{x^3}{6} + \frac{x^2}{2}\right)e^{2x}$$

(2b 30 **pts.)**

$$y'' + y = x\cos x - \cos x$$

Solution: Note that $y_h = C_1\cos x + C_2\sin x.$ Consider

$$y'' + y = -\cos x$$

and

$$v'' + v = -\sin x$$

Let $w = y + iv,$ and consider

$$w'' + w = -e^{ix}$$

Since $p(\lambda) = \lambda^2 + 1$ and $p(i) = 0, p'(\lambda) = 2\lambda,$ so $p'(i) = 2i \neq 0$

$$w_{p_1} = -\frac{xe^{ix}}{2i} = \frac{1}{2}ixe^{ix}$$

Hence

$$y_{p_1} = \operatorname{Re} w_{p_1} = -\frac{x}{2}\sin x$$

Since e^{ix} is a homogeneous solution and xe^{ix} corresponds to a right hand side of $e^{ix},$ we let

$$w_{p_2} = (A_1x + A_2x^2)e^{ix}$$

to deal with a right side of the form xe^{ix}.

$$w_{p_2}' = (A_1 + A_2x)e^{ix} + i(A_1x + A_2x^2)e^{ix}$$
$$w_{p_2}'' = 2A_2e^{ix} + 2i(A_1 + 2A_2x)e^{ix} - (A_1x + A_2x^2)e^{ix}$$

Substituting into the DE leads to

$$2A_2e^{ix} + 2i(A_1 + 2A_2x)e^{ix} = xe^{ix}$$

Therefore

$$2A_2 + 2iA_1 = 0$$
$$4iA_2 = 1 \text{ or } A_2 = \frac{1}{4i} = -\frac{i}{4}$$

Then

$$A_1 = -\frac{1}{i}A_2 = \frac{1}{4}$$

$$w_{p_2} = \frac{1}{4}xe^{ix} - \frac{i}{4}x^2e^{ix}$$

$$y_{p_2} = \operatorname{Re} w_{p_2} = \frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x$$

Thus

$$y = y_h + y_{p_1} + y_{p_2} = C_1\cos x + C_2\sin x - \frac{x}{2}\sin x + \frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x$$

(3) Consider the equation

$$x^2y'' + x(x-2)y' + (x^2+2)y = 0$$

which has a regular singular point at $x = 0$.

(3a 10 **pts.)** Find the indicial equation and solve it.

Solution: The DE becomes

$$y'' + \left(1 - \frac{2}{x}\right)y' + \left(1 + \frac{2}{x^2}\right)y = 0$$

Therefore $P = (1 - \frac{2}{x}), Q = \left(1 + \frac{2}{x^2}\right)$ so $p_0 = \lim_{x\to 0} xP = -2$ and $q_0 = \lim_{x\to 0} x^2Q = 2.$ Thus the indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 - 3r + 2 = (r-1)(r-2) = 0$$

Therefore $r = 1, 2.$

(3b 35 **pts.)** Find the first **four** nonzero terms in the series solution corresponding to the **larger** root of the

indicial equation.

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$y_1' = \sum_{n=0}^{\infty} a_n(n+2)x^{n+1}$$

$$y_1'' = \sum_{n=0}^{\infty} a_n(n+2)(n+1)x^n$$

Substituting into the DE

$$x^2y'' + x(x-2)y' + (x^2+2)y = 0$$

we have

$$\sum_{n=0}^{\infty} a_n(n+2)(n+1)x^{n+2} + \sum_{n=0}^{\infty} a_n(n+2)x^{n+3} - 2\sum_{n=0}^{\infty} a_n(n+2)x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+4} + 2\sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

We shift the sum with x^{n+3} in it by letting $k+2 = n+3$ or $k = n+1$ and the sum with x^{n+4} in it by letting $k+2 = n+4$ or $k = n+2.$ Then

$$\sum_{n=0}^{\infty} a_n\{(n+2)(n+1) - 2(n+2) + 2\}x^{n+2} + \sum_{k=1}^{\infty} a_{k-1}(k+1)x^{k+2} + \sum_{k=2}^{\infty} a_{k-2}x^{k+2} = 0$$

$$\sum_{n=0}^{\infty} a_n(n)(n+1)x^{n+2} + \sum_{k=1}^{\infty} a_{k-1}(k+1)x^{k+2} + \sum_{k=2}^{\infty} a_{k-2}x^{k+2} = 0$$

$$\sum_{m=2}^{\infty} [a_m(m)(m+1) + a_{m-1}(m+1) + a_{m-2}]x^{m+2} + (2a_1 + 2a_0)x^3 = 0$$

Hence

$$a_1 = -a_0$$

and the recurrence relation is

$$a_m(m)(m+1) + a_{m-1}(m+1) + a_{m-2} = 0 \qquad m = 2, 3, 4, \ldots$$

$m = 2$ yields $6a_2 + 3a_1 + a_0 = 0$ or

$$a_2 = \frac{1}{3}a_0$$

Setting $m = 3$ we have $12a_3 + 4a_2 + a_1 = 0$ or $12a_3 = -\frac{4}{3}a_0 + a_0 = -\frac{1}{3}a_0$. Therefore

$$a_3 = -\frac{1}{36}a_0$$

Finally

$$y_1 = a_0x^2 + a_1x^3 + \cdots = a_0\left[x^2 - x^3 + \frac{1}{3}x^4 - \frac{1}{36}x^5 + \cdots\right]$$

(3c 35 **pts.)** Find a second linearly independent solution of the equation. Give the first **three** nonzero terms

of this solution.

Solution: Since the roots differ by an integer, then

$$y_2 = Ay_1 \ln x + \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$y_2' = Ay_1' \ln x + A\frac{y_1}{x} + \sum_{n=0}^{\infty} c_n(n+1)x^n$$

$$y_2'' = Ay_1'' \ln x + 2A\frac{y_1'}{x} - A\frac{y_1}{x^2} + \sum_{n=1}^{\infty} c_n(n+1)nx^{n-1}$$

Substituting into the DE we have

$$x^2y'' + x(x-2)y' + (x^2+2)y = 0$$

$$A\ln x[x^2y_1'' + x(x-2)y_1' + (x^2+2)y_1] + 2Axy_1' - Ay_1 + \sum_{n=1}^{\infty} c_n(n+1)nx^{n+1} + Axy_1$$
$$+ \sum_{n=0}^{\infty} c_n(n+1)x^{n+2} - 2Ay_1 - 2\sum_{n=0}^{\infty} c_n(n+1)x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+3} + 2\sum_{n=0}^{\infty} c_n x^{n+1}$$
$$= 0$$

Since y_1 is a solution of the DE, then the terms times $\ln x$ vanish. We shift the sum with powers x^{n+2} in it (by letting $k+1 = n+2$ or $k = n+1$) and the sum with powers x^{n+3} in it (by letting $k+1 = n+3$ or $k = n+2$) and get

$$2Axy_1' + A[xy_1 - 3y_1] + \sum_{n=0}^{\infty} c_n\{n(n+1) - 2(n+1) + 2\}x^{n+1} + \sum_{k=1}^{\infty} kc_{k-1}x^{k+1} + \sum_{k=2}^{\infty} c_{k-2}x^{k+1} = 0$$

The parenthesis in the first sum simplifies to $n(n+1)$ so we have

$$2Axy_1' + A[xy_1 - 3y_1] + \sum_{m=2}^{\infty} c_m\{m(m-1)\}x^{m+1} + \sum_{m=1}^{\infty} mc_{m-1}x^{m+1} + \sum_{m=2}^{\infty} c_{m-2}x^{m+1} = 0$$

or

$$2Axy_1' + A[xy_1 - 3y_1] + \sum_{m=2}^{\infty}\{c_m m(m-1) + mc_{m-1} + c_{m-2}\}x^{m+1} + c_0x^2 = 0$$

Let $a_0 = 1$ in the expression for y_1. Then

$$y_1 = x^2 - x^3 + \frac{1}{3}x^4 - \frac{1}{36}x^5 + \cdots$$
$$y_1' = 2x - 3x^2 + \frac{4}{3}x^3 - \frac{5}{36}x^4 + \cdots$$

and the last equation becomes

$$2xA\left[2x - 3x^2 + \frac{4}{3}x^3 - \frac{5}{36}x^4 + \cdots\right] + A\left[x^3 - x^4 + \frac{1}{3}x^5 - \frac{1}{36}x^6 + \cdots - 3x^2 + 3x^3 - x^4 + \frac{1}{12}x^5 + \cdots\right]$$
$$+ \sum_{m=2}^{\infty}\{c_m m(m-1) + mc_{m-1} + c_{m-2}\}x^{m+1} + c_0x^2$$
$$= 0$$

Then for the coefficients of x^2 we have

$$4A - 3A + c_0 = 0 \quad \text{or} \quad A = -c_0$$

Thus we have

$$-2xc_0\left[2x - 3x^2 + \frac{4}{3}x^3 - \frac{5}{36}x^4 + \cdots\right] - c_0\left[x^3 - x^4 + \frac{1}{3}x^5 - \frac{1}{36}x^6 + \cdots - 3x^2 + 3x^3 - x^4 + \frac{1}{12}x^5 + \cdots\right]$$
$$+ \sum_{m=2}^{\infty}\{c_m m(m-1) + mc_{m-1} + c_{m-2}\}x^{m+1} + c_0x^2$$
$$= 0$$

For the coefficients of x^3 we have

$$6c_0 - 4c_0 + 2(1)c_2 + 2c_1 + c_0 = 0$$

or

$$2c_2 = -3c_0 - 2c_1$$

or

$$c_2 = \frac{-3}{2}c_0 - c_1$$

For the coefficients of x^4 we have

$$-\left[\frac{8}{3} - 2\right]c_0 + 3(2)c_3 + 3c_2 + c_1 = 0$$

so

$$6c_3 = -3c_2 - c_1 + \frac{2}{3}c_0$$

and

$$6c_3 = \frac{9}{2}c_0 + 3c_1 - c_1 + \frac{2}{3}c_0 = \frac{31}{6}c_0 + 2c_1$$

Thus

$$c_3 = \frac{31}{36}c_0 + \frac{1}{3}c_1$$

$$\begin{aligned} y_2 &= Ay_1 \ln x + \sum_{n=0}^{\infty} c_n x^{n+1} \\ &= c_0\left[-y_1 \ln x + x - \frac{3}{2}x^3 + \frac{31}{36}x^4 + \cdots\right] + c_1\left[x^2 - x^3 + \frac{1}{3}x^4 + \cdots\right] \\ &= c_0\left[-y_1 \ln x + x - \frac{3}{2}x^3 + \frac{31}{36}x^4 + \cdots\right] + c_1 y_1 \end{aligned}$$

Therefore the second solution is

$$y_2 = c_0^*\left[y_1 \ln x - x + \frac{3}{2}x^3 - \frac{31}{36}x^4 + \cdots\right]$$