Ma 530

Partial Differential Equations - Separation of Variables in Multi-Dimensions

Temperature in an Infinite Cylinder

Consider an infinitely long, solid, circular cylinder of radius c with its axis coinciding with the $z-$axis of a system of cylindrical coordinates $r, \theta, z.$ We assume that the surface of the cylinder $r = c$ is kept at zero temperature. If $u = u(x,y,z,t)$ is the temperature at any point (x,y,z) at any time $t,$ then it can be shown that u satisfies the three dimensional heat equation

$$u_{xx} + u_{yy} + u_{xx} = \frac{1}{k} u_t \qquad (1a)$$

where $k > 0$ is a constant.

Suppose at time $t = 0$ the temperature u at any point in the cylinder is a function of r alone, i.e.

$$u(x,y,z,0) = f(r)$$

Since we are dealing with a cylinder, we shall use cylindrical coordinates. Then

$$u(x,y,z,t) = u(r,\theta,z,t)$$

It may be shown that since u does not initially depend on θ and $z,$ then it will not depend on them for all time, i.e., $u(r,\theta,z,t) = u(r,t)$ In cylindrical coordinates the heat equation $(1a)$ becomes

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz} = \frac{1}{k} u_t$$

Since $u = u(r,t)$ we have

$$u_{rr} + \frac{1}{r} u_r = \frac{1}{k} u_t \qquad (1b)$$

$$u(r,0) = f(r) \qquad (2)\ \text{IC}$$

$$u(c,t) = 0 \qquad (3)\ \text{BC}$$

We shall solve $(1a,2,3)$ using separation of variables. We assume the $u(r,t) = R(r)T(t)$ and $(1b)$ implies

$$R''T + \frac{1}{r} R'T = \frac{1}{k} RT'$$

$$\frac{R''}{R} + \frac{R'}{rR} = \frac{1}{k}\frac{T'}{T} = \text{constant}$$

Since physically we know that the temperature must go to 0 as $t \to \infty,$ we choose the constant so that T decays with time. Thus

$$\frac{R''}{R} + \frac{R'}{rR} = \frac{1}{k}\frac{T'}{t} = -\lambda^2$$

We are then lead to the two ordinary differential equations

$$T' + k\lambda^2 T = 0$$
$$rR'' + R' + \lambda^2 rR = 0$$

Then

$$T(t) = T_0 e^{-k\lambda^2 t}$$

where T_0 is a constant.

It turns out that the equation for R is a form of Bessel's equation. To see this let $s = \lambda r.$ Then

$$\frac{dR}{dr} = \frac{dR}{ds}\frac{ds}{dr} = \lambda\frac{dR}{ds}$$
$$\frac{d^2R}{ds^2} = \lambda^2\frac{d^2R}{ds^2}$$

and the equation above for R becomes

$$s\frac{d^2R}{ds^2} + \frac{dR}{ds} + sR = 0 \tag{4}$$

Equation (4) is Bessel's equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

with $p = 0.$ The solution of (4)

$$R(s) = c_1 J_0(s) + c_2 Y_0(s)$$

where J_0 and Y_0 are the Bessel functions of order zero of the first and second kind respectively. Since the solution at $r = 0$ must be finite and since Y_0 is infinite at 0, we set $c_2 = 0$ so that

$$R(r) = c_0 J_0(\lambda r)$$

Therefore

$$u(r,t) = AJ_0(\lambda r)e^{-k\lambda^2 t}$$

The boundary condition (3) implies that

$$J_0(\lambda c) = 0$$

Thus λ is such that λc is a root (zero) of $J_0(x).$ Since J_0 has an infinite number of roots, then $\lambda = \lambda_n$ where $\lambda_n c$ is the nth zero of $J_0.$ Thus

$$u_n(r,t) = A_n J_0(\lambda_n r)e^{-k\lambda_n^2 t} \quad n = 1,2,\ldots$$

The functions $J_0(\lambda_n r)$ form an infinite set of eigenfunctions. It can be shown that these functions form a complete, orthogonal set with respect to the weight function $r.$ Thus to satisfy condition (2) $u(r,0) = f(r)$ we let

$$u(r,t) = \sum_{n=1}^{\infty} u_n(r,t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)e^{-k\lambda_n^2 t}$$

so that

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)$$

and

$$A_n = \frac{2}{c^2[J_1(\lambda_n c)]^2}\int_0^c rf(r)J_0(\lambda_n r)dr$$

A Vibrating Circular Membrane

The motion of a membrane attached to a frame in the x,y –plane is governed by the two dimensional wave equation. We assume the membrane to be thin and flexible with a constant tension $\overline{T}$, per unit length across any line acting tangentially to the membrane at each point. Also we assume that the mass per unit area, δ is constant and that the displacement $z(x,y,t)$ of the membrane is small. If no external forces act on the membrane, then the transverse displacement z satisfies the two-dimensional wave equation

$$z_{xx} + z_{yy} = \frac{1}{a^2} z_{tt}$$

where $a^2 = \overline{T}/\delta$.

Consider now the vibrations of a circular membrane of radius c with center at the origin of the x,y –plane. If the outer edge is to be kept fixed, then $z = 0$ on $x^2 + y^2 = c^2$. This boundary condition in unwieldy in rectangular coordinates. since our membrane is circular, we shall switch to polar coordinates r and θ where $x = r\cos\theta,\ y = r\sin\theta.$ Then $z = z(r,\theta,t)$ and in polar coordinates the wave equation becomes

$$z_{rr} + \frac{1}{r} z_r + \frac{1}{r^2} z_{\theta\theta} = \frac{1}{a^2} z_{tt} \qquad t > 0,\ \ 0 \le r \le c,\ \ -\pi \le \theta \le \pi \tag{1}$$

The boundary condition along the edge of the circular membrane takes the form

$$z(c,\theta,t) = 0 \quad -\pi \le \theta \le \pi \tag{2}$$

Equation (2) is clearly a much better form than the form of this condition in rectangular coordinates. In addition to the boundary condition (2) one must be given initial conditions. We assume that the initial displacement of the membrane is $f(r,\theta)$ and that it is released from rest. Hence we have

$$z(r,\theta,0) == f(r,\theta) \tag{3a}$$

$$z_t(r,\theta,0) = 0 \tag{3b}$$

We again assume that the problem can be solved by separation of variables and seek a solution

$$z(r,\theta,t) = R(r)S(\theta)T(t)$$

Equation (1) then implies

$$\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{1}{r^2}\frac{S''(\theta)}{S(\theta)} = \frac{T''(t)}{a^2T(t)} \tag{4}$$

Since the left hand side of (4) is independent of t and the right hand side is independent of θ and r, we have

$$\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{1}{r^2}\frac{S''(\theta)}{S(\theta)} = \frac{T''(t)}{a^2T(t)} = k,\ \ k \text{ a constant}$$

It is to be expected that the motion will be periodic in time. Thus we expect $T(t)$ to be expressed in terms of sines and cosines. To get this we let $k = -\lambda^2.$ This leads to

$$T'' + a^2\lambda^2 T = 0$$

Condition $(3b)$ implies $T'(0) = 0,$ so

$$T(t) = A\cos a\lambda t$$

Equation (4) also tells us that

$$\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{1}{r^2}\frac{S''(\theta)}{S(\theta)} = -\lambda^2$$

or

$$\frac{r}{R}(rR'' + R') + \lambda^2 r^2 = -\frac{S''(\theta)}{S(\theta)} \tag{5}$$

Since the left hand side is independent of θ and the right hand side is independent of $r,$ both sides must equal a constant. Since we are using polar coordinates, $S(\theta)$ must be expressed as sines and cosines. Hence

$$-\frac{S''(\theta)}{S(\theta)} = \mu^2 \quad \text{or} \quad S'' + \mu^2 S = 0$$

which leads to

$$S(\theta) = C_1 \cos\mu\theta + C_2 \sin\mu\theta$$

The displacement z must be a single-valued function of $\theta.$ This means that $S(\theta)$ must be periodic with period $2\pi.$ Thus μ must be an integer $n,$ and

$$S_n(\theta) = c_n \cos n\theta + d_n \sin n\theta \quad n = 1,2,3,\ldots$$

Equation (5) implies

$$\frac{r}{R}(rR'' + R') + \lambda^2 r^2 = n^2$$

or

$$r^2 R'' + rR' + (\lambda^2 r^2 - n^2)R = 0$$

This is Bessel's equation and it has the solutions

$$R(r) = C_3 J_n(\lambda r) + C_4 Y_n(\lambda r) \quad n = 0,1,2,\ldots$$

Since the displacement at the origin must be finite, we set $C_4 = 0,$ since $\lim_{r\to 0} Y_n(\lambda r) = -\infty.$ Thus

$$R(r) = D_n J_n(\lambda r) \quad n = 0,1,2,\ldots$$

The condition $(3b)$ implies $R(c) = 0,$ so λ must be such that

$$J_n(\lambda c) = 0 \quad n = 0,1,2,\ldots$$

Hence λc is one of the zeroes of $J_n,$ $n = 0,1,2,\ldots$ that is, $\lambda = \lambda_{nk}$ where λ_{nk} is the kth zero of the nth Bessel function $J_n.$ Thus

$$R_{nk}(r) = D_{nk} J_n(\lambda_{nk} r) \quad n = 0,1,2,\ldots; \;\; k = 1,2,3,\ldots$$

This means that

$$z_{nk}(r,\theta,t) = J_n(\lambda_{nk} r)[a_{nk}\cos n\theta + b_{nk}\sin n\theta]\cos a\lambda_{nk} t \quad n = 0,1,2,\ldots; \;\; k = 1,2,3,\ldots$$

Thus we have a double infinity of eigenvalues and eigenfunctions.

We must still satisfy the condition

$$z(r,\theta,0) = f(r,\theta)$$

Since f depends on two variables we assume that $f(r,\theta)$ has a Fourier series expansion of the form

$$f(r,\theta) = \sum_{n=0}^{\infty}[A_n(r)\cos n\theta + B_n(r)\sin n\theta]$$

where

$$A_0(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(r,\theta)dr \tag{6a}$$

$$A_n(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(r,\theta)\cos n\theta dr \quad n = 1,2,3,\ldots \tag{6b}$$

$$B_n(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(r,\theta)\sin n\theta dr \quad n = 1,2,3,\ldots \tag{6c}$$

We let

$$z(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} J_n(\lambda_{nk}r)[a_{nk}\cos n\theta + b_{nk}\sin n\theta]\cos a\lambda_{nk}t \tag{7}$$

Then

$$z(r,\theta,0) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} J_n(\lambda_{nk}r)[a_{nk}\cos n\theta + b_{nk}\sin n\theta] = f(r,\theta) = \sum_{n=0}^{\infty}[A_n(r)\cos n\theta + B_n(r)\sin n\theta]$$

Therefore

$$A_n(r) = \sum_{k=1}^{\infty} a_{nk}J_n(\lambda_{nk}r)$$

$$B_n(r) = \sum_{k=1}^{\infty} b_{nk}J_n(\lambda_{nk}r)$$

Since $A_n(r)$ and $B_n(r)$ are known via $(6a,b,c)$, we see that the a_{nk} and b_{nk} are the Fourier-Bessel coefficients in the expansions of A_n and B_n. Therefore

$$a_{nk} = \frac{2}{c^2[J_{n+1}(\lambda_{nk}c)]^2}\int_0^c rA_n(r)J_n(\lambda_{nk}r)dr$$

$$b_{nk} = \frac{2}{c^2[J_{n+1}(\lambda_{nk}c)]^2}\int_0^c rB_n(r)J_n(\lambda_{nk}r)dr$$

Thus z is given by (7), with the coefficients determined by the above formulas.

Heat in a Hollow Cylinder

Suppose a homogeneous hollow cylinder which occupies the region $a \le r \le b, 0 \le \theta \le 2\pi, 0 \le z \le h$ (that is a piece of pipe of length h and thickness $b - a$) has its ends $z = 0$ and $z = h$ maintained at temperatures 0° and 100° respectively. The faces at $r = a$ and $r = b$ are insulated against the flow of heat. Let $\phi(r,\theta,z,t)$ the temperature in the solid. Assuming an initial temperature distribution $f(r,\theta,z)$, we must solve the boundary value problem for $\phi(r,\theta,z,t)$ in the solid.

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} + \phi_{zz} = \frac{1}{c^2}\phi_t$$

or in cylindrical coordinates

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} + \phi_{zz} = \frac{1}{c^2}\phi_t \tag{1}$$

$$\phi(r,\theta,0,t) = 0 \quad \phi(r,\theta,h,t) = 100 \quad \text{boundary conditions} \tag{2a, 2b}$$

$$\phi_r(a,\theta,z,t) = \phi_r(b,\theta,z,t) = 0 \quad \text{insulation condition} \tag{3a,b}$$

$$\phi(r,\theta,z,0) = f(r,\theta,z) \quad \text{initial condition} \tag{4}$$

Notice that the boundary condition $(2b)$ is nonhomogeneous. To solve the problem we proceed as follows: Let

$$\phi = \phi^S + \phi^T$$

where ϕ^S is the *steady-state solution* that does not depend on t. Hence

$$\nabla^2\phi^S = 0$$

$$\phi^S(r,\theta,0) = 0 \quad \phi^S(r,\theta,h) = 100$$

$$\phi_r^S(a,\theta,z) = \phi_r^S(b,\theta,z) = 0$$

Remark: $\nabla^2\phi^S = 0$ is Laplace's equation which governs steady state temperature problems.

Now ϕ^T is the *transient solution* and satisfies

$$\nabla^2\phi^T = \frac{1}{c^2}\phi_t^T$$

$$\phi^T(r,\theta,0,t) = 0 \quad \phi^T(r,\theta,h,t) = 0$$

$$\phi_r^T(a,\theta,z,t) = \phi_r^T(b,\theta,z,t) = 0$$

Note that the boundary conditions for ϕ^T are both homogeneous. Thus $\phi^S + \phi^T$ satisfies $(1,2,3,4)$. The initial condition for ϕ^T is

$$\phi^T(r,\theta,z,0) = f(r,\theta,z) - \phi^S(r,\theta,z)$$

The solution to the problem for ϕ^S is just the steady state solution of uniform flow of heat from the high temperature end to the low temperature end and hence

$$\phi^S = 100\frac{z}{h}$$

For the transient solution we use separation of variables. Let

$$\phi^T = U(r)V(\theta)W(z)F(t)$$

Then (1) implies

$$U''VWF + \frac{1}{r}U'VWF + \frac{1}{r^2}UV''WF + UVW''F = \frac{1}{c^2}UVWF'$$

or

$$\frac{U''}{U} + \frac{1}{r}\frac{U'}{U} + \frac{1}{r^2}\frac{V''}{V} + \frac{W''}{W} = \frac{1}{c^2}\frac{F'}{F} = -\lambda^2 \quad \lambda > 0$$

Thus

$$F' + \lambda^2c^2F = 0$$

so

$$F = c_1e^{-\lambda^2c^2t}$$

We also have

$$\frac{U''}{U} + \frac{1}{r}\frac{U'}{U} + \frac{1}{r^2}\frac{V''}{V} = -\lambda^2 - \frac{W''}{W} = -\alpha^2 \tag{5}$$

(We shall later why the constant is chosen as $-\alpha^2$.) Then we have

$$\frac{W''}{W} = -\lambda^2 + \alpha^2 \quad \text{or} \quad W'' + (\lambda^2 - \alpha^2)W = 0$$

The homogeneous boundary conditions at $z = 0$ and $z = h$ imply

$$W(0) = W(h) = 0$$

so

$$W(z) = c_2 \cos\sqrt{\lambda^2 - \alpha^2}\, z + c_3 \sin\sqrt{\lambda^2 - \alpha^2}\, z$$

Applying the conditions on W yields $c_2 = 0$ and

$$\lambda^2 - \alpha^2 = \frac{k^2\pi^2}{h^2} \quad k = 1, 2, \ldots$$

Hence

$$W(z) = c_3 \sin\left(\frac{k^2\pi^2}{h^2}\right)$$

Equation (5) implies

$$\frac{U''}{U} + \frac{1}{r}\frac{U'}{U} + \frac{1}{r^2}\frac{V''}{V} = -\alpha^2$$

or

$$r^2\frac{U''}{U} + r\frac{U'}{U} + \alpha^2 r^2 = -\frac{V''}{V} = n^2$$

We set $\frac{V''}{V} = -n^2$ since ϕ^T must be a single-valued function of θ. Thus

$$V'' + n^2 V = 0$$

and

$$V(\theta) = c_4 \cos n\theta + c_5 \sin n\theta \quad n = 0, 1, 2, \ldots$$

The equation for U is

$$r^2 U'' + rU' + (\alpha^2 r^2 - n^2)U = 0$$

(The choice of $-\alpha^2$ above was motivated by the desire to get this last equation which is Bessel's equation.)

Therefore,

$$U(r) = c_6 J_n(\alpha r) + c_7 Y_n(\alpha r)$$

where $Y_n(\alpha r)$ is the Bessel function of the second kind.
The conditions about insulation, namely,

$$\phi_r^T(a, \theta, z, t) = \phi_r^T(b, \theta, z, t) = 0$$

must be satisfied. These conditions imply

$$c_6 J_n'(\alpha a) + c_7 Y_n'(\alpha a) = 0$$
$$c_6 J_n'(\alpha b) + c_7 Y_n'(\alpha b) = 0$$

These two equations are to be solved to yield nontrivial solutions for c_6 and c_7. This means that α is a solution of

$$J_n'(\alpha a) Y_n'(\alpha b) - J_n'(\alpha b) Y_n'(\alpha a) = 0$$

For each fixed n there are an infinite number of zeroes α. Thus $\alpha = \alpha_{mn}$ where α_{mn} is the mth root of the nth equation ($m = 1, 2, \ldots, n = 0, 1, 2, \ldots$).

If α_{mn} is a root of the above equation, then

$$c_7 = -c_6 \frac{J_n'(\alpha_{mn} a)}{Y_n'(\alpha_{mn})}$$

and

$$\begin{aligned} U(r) &= c_6 \left[J_n(\alpha_{mn} r) - Y_n(\alpha_{mn} r) \frac{J_n'(\alpha_{mn} a)}{Y_n'(\alpha_{mn})} \right] \\ &= c_6 \left[\frac{J_n(\alpha_{mn} r) Y_n'(\alpha_{mn}) - Y_n(\alpha_{mn} r) J_n'(\alpha_{mn} a)}{Y_n'(\alpha_{mn} a)} \right] \\ &= c_6 U_{mn}(r) \end{aligned}$$

Thus the formal solution is

$$\phi^T(r, \theta, z, t) = \sum_{n=o}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} [a_{kmn} \cos n\theta + b_{kmn} \sin n\theta] \sin\left(\frac{k\pi}{h} z \right) U_{mn}(r) e^{-c^2 \lambda_{kmn}^2 t}$$

Note that λ depends on k and α m and α depends on n and n so $\lambda = \lambda_{kmn}$. Thus

$$\phi(r, \theta, z, t) = 100 \frac{z}{h} + \phi^T$$

To satisfy the initial condition we must have

$$f(r, \theta, z) = \sum_{n=o}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} [a_{kmn} \cos n\theta + b_{kmn} \sin n\theta] \sin\left(\frac{k\pi}{h} z \right) U_{mn}(r)$$

It can be shown that the functions

$$\begin{aligned} \psi_{kmn}^{(1)} &= \cos n\theta \sin\left(\frac{k\pi}{h} z \right) U_{mn} \\ \psi_{kmn}^{(2)} &= \sin n\theta \sin\left(\frac{k\pi}{h} z \right) U_{mn} \end{aligned}$$

form two orthogonal sets. These may be used to determine the constants a_{kmn} and b_{kmn}.