Ma 530

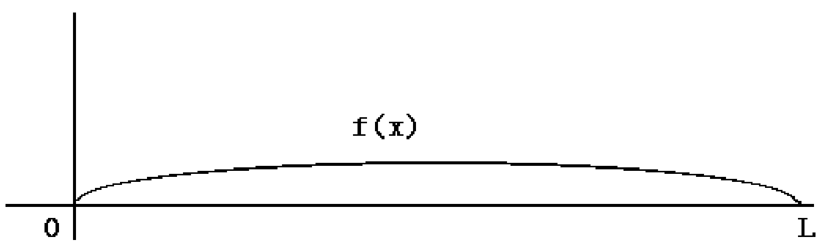
Partial Differential Equations

The Vibrating String

It may be shown that the equation governing a string of length L vibrating is

$$y_{xx}(x,t) = \frac{\partial^2 y}{\partial x^2} = \frac{1}{\alpha^2} y_{tt}(x,t) \qquad (1)$$

$y(x,t)$ is the displacement of the string from the x –axis at the point x at time t. Equation (1) is called the wave equation. α is a constant. Suppose string is held fixed at the ends $x = 0$ and $x = L$



$\Rightarrow$

$$(2a) \quad y(0,t) = 0 \qquad t \geq 0 \qquad B.C.$$

$$(2b) \quad y(L,t) = 0 \qquad t \geq 0 \qquad B.C.$$

Also suppose at $t = 0$ string has displacement $y = f(x)$ and is released from rest

$\Rightarrow$

$$(3a) \quad y(x,0) = f(x) \qquad 0 \leq x \leq L \qquad I.C.$$

$$(3b) \quad y_t(x,0) = 0 \qquad 0 \leq x \leq L \qquad I.C.$$

In order to solve the above problem we shall assume $y(x,t) = X(x)T(t)$ separation of variables $\Rightarrow y_x = X'T \quad y_{xx} = X''T \quad y_{tt} = XT''$. Note that $X', T', \ldots$ are ordinary derivatives of X with respect to x and T with respect to t. Now the P.D.E. (1)

$\Rightarrow$

$$X''T = \frac{1}{\alpha^2}XT''$$

$\Rightarrow$

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T''}{T}$$

.

Note that the left hand side is a function of x only, whereas the right hand side is a function of t only. This implies that each side must equal the same constant. Therefore

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T''}{T} = k$$

Hence we get the two ordinary differential equations

$$X'' - kX = 0 \qquad \text{and} \qquad T'' - \alpha^2 kT = 0$$

Now $y(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0,$ whereas $y(L,t) = X(L)T(t) = 0 \Rightarrow X(L) = 0.$ Therefore we must solve the problem

$$X'' - kX = 0 \qquad X(0) = X(L) = 0.$$

There are three cases. If $k = 0 \Rightarrow X \equiv 0$. If $k > 0 \Rightarrow X = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}$.
and the boundary conditions $\Rightarrow c_1 = c_2 = 0$.

For the case $k < 0,$ let $k = -\lambda^2$
$\Rightarrow$

$$X'' + \lambda^2 X = 0 \qquad X(0) = X(L) = 0$$

This is an eigenvalue problem. The solution to the DE is

$$X = c_1 \sin \lambda x + c_2 \cos \lambda x$$

$X(0) = 0 \Rightarrow c_2 = 0$ whereas $X(L) = 0 \quad \Rightarrow c_1 \sin \lambda L = 0 \quad \Rightarrow \lambda = \frac{n\pi}{L}$ for $n = \pm 1, \pm 2, \pm 3, \ldots.$
Since $\sin(-x) = -\sin x$ we may disregard the negative values of n.
Therefore

$$X(x) = c_n \sin \frac{n\pi}{L}x \qquad n = 1,2,3,\ldots$$

For $T(t)$ we have the equation

$$T'' + \lambda^2 T = 0,$$

since $k = -\lambda^2$. Thus

$$T = c\sin\alpha\lambda t + d\cos\alpha\lambda t = a_n \sin\frac{n\pi\alpha}{L}t + b_n\cos\frac{n\pi\alpha t}{L}.$$

But $y_t(x,0) = X(x)T'(0) = 0 \Rightarrow T'(0) = 0.$ Now

$$T'(t) = a_n\left(\frac{n\pi}{L}\right)\cos\frac{n\pi}{L}t - b_n\left(\frac{n\pi}{L}\right)\sin\frac{n\pi t}{L}$$

, so $T'(0) = 0 \Rightarrow a_n = 0$ for all n .

Therefore

$$T(t) = b_n\cos\frac{n\pi\alpha t}{L},\ \ n = 1,2,\ldots$$

and we have finally that

$$y_n(x,t) = X(x)T(t) = c_n\sin\frac{n\pi x}{L} \times b_n\cos\frac{n\pi\alpha t}{L}$$

Let $c_n \times b_n = d_n$.

We note that

$$y_n(x,t) = d_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi\alpha t}{L}\right)$$

satisfies the P.D.E. $y_{xx} = \frac{1}{\alpha^2}y_{tt}$ (1) and the boundary conditions $y(0,t) = y(L,t) = 0$ $(2a,2b),$ as well as the initial condition $y_t(x,0) = 0$ $(3b)$.

What about the condition $y(x,0) = f(x)$? Notice that

$$y(x,t) = \sum_1^\infty d_n\sin\frac{n\pi x}{L}\cos\frac{n\pi\alpha t}{L}$$

is also a solution since of $(1), (2a,b)$ and $(3b)$. Thus $y(x,t)$ is solution of everything except condition $(3a),$ namely, $y(x,0) = f(x)$.

But

$$y(x,0) = \sum_1^\infty d_n\sin\frac{n\pi x}{L} = f(x).$$

Therefore if f has a Fourier sine series expansion we let

$\Rightarrow$

$$d_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}.$$

Now with these coefficients d_n

$$y(x,t) = \sum_1^\infty d_n \sin\frac{n\pi x}{L}\cos\frac{n\pi\alpha t}{L}$$

is a solution to entire problem $(1),(2a,2b),(3a,3b)$.

Example

$$y_{xx} = y_{tt} \qquad y(0,t) = y(L,t) = 0$$
$$y_t(x,0) = 0$$
$$y(x,0) = -2\sin\frac{\pi x}{L}$$

Now

$$y(x,t) = \sum_1^\infty d_n \sin\frac{n\pi x}{L}\cos\frac{n\pi t}{L}$$

$$d_n = \frac{2}{L}\int_0^L (-2)\sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}dx = 0 \qquad n = 2,3,\ldots$$

$$\begin{aligned} d_1 &= \frac{2}{L}\int_0^L (-2)\sin^2\frac{n\pi x}{L}dx \\ &= \frac{-4}{L}\left[\int_0^L \frac{1-\cos(2\frac{n\pi x}{L})}{2}\right]dx = \frac{-4}{L}\left[\frac{x}{2} - \frac{\sin\left(2\frac{n\pi x}{L}\right)}{\frac{2n\pi}{L}}\right]_0^L = -2 \end{aligned}$$

Another way to get d_1 is to note that

$$y(x,0) = \sum_1^\infty d_n\sin\frac{n\pi x}{L} = d_1\sin\frac{\pi x}{L} + d_2\sin\left(\frac{2\pi x}{L}\right) + \cdots = -2\sin\frac{\pi x}{L}$$

$\Rightarrow$ solution is

$$y(x,t) = -2\sin\frac{\pi x}{L}\cos\frac{\pi t}{L}.$$

Example Use separation of variables, $u(r,\theta) = R(r)T(\theta)$, to find ordinary differential equations which $R(r)$ and $T(\theta)$ must satisfy if $u(r,\theta)$ is to be a solution of

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Do not solve these equations.

Solution: Let $u(r,\theta) = R(r)T(\theta)$ then $\quad u_r = R'(r)T(\theta) \quad u_{rr} = R''(r)T(\theta) \quad u_{\theta\theta} = R(r)T''(\theta)$ and $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ becomes

$$R''(r)T(\theta) + \frac{1}{r}R'(r)T(\theta) + \frac{1}{r^2}R(r)T''(\theta) = 0$$

$$r^2R''(r)T(\theta) + rR'(r)T(\theta) = -R(r)T''(\theta)$$

$$\frac{r^2R''(r) + rR'(r)}{-R(r)} = \frac{T''(\theta)}{T(\theta)} = k$$

since R and T are independent
resulting in the equations

$$r^2R''(r) + rR'(r) + kR(r) = 0$$

and

$$T''(\theta) - kT(\theta) = 0$$

Example Solve

PDE $\quad u_{xx} = 4u_{tt}$

BCS $\quad u_x(0,t) = 0 \qquad u_x(\pi,t) = 0$

ICs $\quad u(x,0) = 0 \qquad u_t(x,0) = -9\cos(4x) + 16\cos(8x)$

Derive the solution. The solution should not have any arbitrary constants in it.

Solution:

Let $u(x,t) = X(x)T(t)$. Then differentiating and substituting in the PDE yields

$$X''T = 4XT''$$
$$\Rightarrow$$
$$\frac{X''}{X} = 4\frac{T''}{T}$$

Using the argument that the left hand side is purely a function of x and the right hand side is purely a function of t, and the only way that they can be equal is if they are equal to a constant, we get

$$\frac{X''}{X} = 4\frac{T''}{T} = k \qquad k \text{ a constant}$$

This yields the two *ordinary differential equations*

$$X'' - kX = 0 \qquad \text{and} \qquad T'' - \frac{1}{4}kT = 0$$

The boundary condition $u_x(0,t) = 0$ implies, since $u_x(x,t) = X'(x)T(t)$ that $X'(0)T(t) = 0.$ We cannot have $T(t) = 0,$ since this would imply that $u(x,t) = 0.$ Thus $X'(0) = 0.$ Similarly, the boundary condition $u_x(\pi,t) = 0$ leads to $X'(\pi) = 0.$

We now have the following boundary value problem for $X(x)$:

$$X'' - kX = 0 \qquad X'(0) = X'(\pi) = 0$$

For $k > 0,$ the only solution is $X = 0.$ For $k = 0$ we have $X = Ax + B.$ $X'(x) = A,$ so the BCs imply that

$$X(x) = B, \quad B \neq 0$$

is a nontrivial solution corresponding to the eigenvalue $k = 0.$
For $k < 0,$ let $-k = \alpha^2,$ where $\alpha \neq 0.$ Then we have the equation

$$X'' + \alpha^2 X = 0$$

and

$$X(x) = c_1 \sin\alpha x + c_2 \cos\alpha x$$

$$X'(x) = c_1\alpha\cos\alpha x - c_2\alpha\sin\alpha x$$

$$X'(0) = c_1\alpha = 0$$

so $c_1 = 0.$

$$X'(\pi) = -c_2\alpha\sin\alpha\pi = 0$$

Therefore $\alpha = n,\ n = 1,2,\ldots$ and the solution is

$$k = -n^2 \qquad X_n(x) = a_n\cos nx \qquad n = 1,2,3,\ldots$$

The case $k = 0$ implies that the equation for T becomes $T'' = 0,$ so $T = At + B.$ The initial condition $u(x,0) = 0$ implies $X(x)T(0) = 0$ so that $T(0) = 0.$ Thus $B = 0$ and $T = At$ for $k = 0.$
Substituting the values of $k = -n^2$ into the equation for $T(t)$ leads to

$$T'' + \frac{n^2}{4}T = 0$$

which has the solution

$$T_n(t) = B_n\sin\frac{nt}{2} + C_n\cos\frac{nt}{2}, \quad n = 1,2,3,\ldots$$

The initial condition $u(x,0) = 0$ implies $X(x)T(0) = 0$ so that $T(0) = 0.$ Thus $C_n = 0.$

We now have the solutions

$$u_n(x,t) = A_n\cos nx\sin\frac{nt}{2} \qquad n = 1,2,3,\ldots$$

$$u_0(x,t) = A_0 t$$

Since the boundary conditions and the equation are linear and homogeneous, it follows that

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = A_0 t + \sum_{n=1}^{\infty} A_n \cos nx \sin \frac{nt}{2}$$

satisfies the PDE, the boundary conditions, and the first initial condition. Since

$$u_t(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \left(\frac{n}{2}\right) \cos nx \cos \frac{nt}{2}$$

the last initial condition leads to

$$u_t(x,0) = -9\cos(4x) + 16\cos(8x) = A_0 + \sum_{n=1}^{\infty} A_n \left(\frac{n}{2}\right) \cos nx.$$

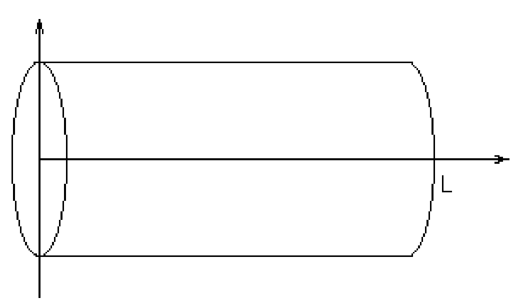
Matching the cosine terms on both sides of this equation leads to

$A_4\left(\frac{4}{2}\right) = -9$ so that $A_4 = -\frac{9}{2}$ and $A_8\left(\frac{8}{2}\right) = 16$ so that $A_8 = 4$. All of the other constants must be zero, since there are no cosine terms or constant terms on the left to match with. Thus

$$u(x,t) = -\frac{9}{2}\cos 4x \sin 2t + 4\cos 8x \sin 4t$$

The Heat Equation

Consider a cylinder parallel to x –axis



Let u denote the temperature in the cylinder. Suppose the ends $x = 0$ and $x = L$ are kept at zero temperature whereas at $t = 0$ the initial temperature distribution is $u = f(x)$. It may be shown that $u = u(x,t)$ satisfies the P.D.E.

$$u_{xx} = \frac{1}{k} u_t \qquad 0 < x < L, \qquad t > 0, \qquad (1)$$

where k is a constant and $k > 0$.

.

Equation (1) is called the heat equation. The physical conditions of the problem imply

$$B.C.\ u(0,t) = 0 = u(L,t) \qquad t \geq 0 \qquad (2)$$

$$I.C.\ u(x,0) = f(x) \qquad 0 \leq x \leq L \qquad (3)$$

We want to determine $u(x,t)$, i.e. the temperature in the cylinder at any point x at any time t. Again we use separation of variables. The assumption $u(x,t) = X(x)T(t)$ leads to

$$\frac{X''(x)}{X(x)} = \frac{1}{k}\frac{T'(t)}{T(t)} = -\lambda^2$$

$\Rightarrow X'' + \lambda^2 X = 0 \qquad X(0) = X(L) = 0$ and $T' + k\lambda^2 T = 0$.

$\Rightarrow X_n = c_n \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, \ldots \qquad \lambda_n = \frac{n\pi}{L} \Rightarrow$

$$T' + k\frac{n^2\pi^2}{L^2} T = 0$$

$\Rightarrow$

$$T_n(t) = d_n e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

$\Rightarrow$

$$u_n(x,t) = a_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \sin\frac{n\pi x}{L}$$

satisfies (1) and (2) $\Rightarrow$

$$u_n(x,t) = \sum_1^\infty a_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \sin\frac{n\pi x}{L}$$

also satisfies (1) and (2).

We need to satisfy (3) namely, $u(x,0) = f(x)$ However,

$$u(x,0) = \sum_{1}^{\infty} a_n \sin \frac{n\pi x}{L}$$

Thus we take a_n to be the Fourier sine coefficients of $f(x)$. Hence

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

Remark. The factor $e^{-\left(\frac{n\pi}{L}\right)^2 kt} \to 0$ as $t \to \infty \Rightarrow \lim_{t\to\infty} u(x,t) = 0$ as expected from the physical problem.

Example Solve:

P.D.E.: $u_{xx} = 4u_t$

B.C.s: $u(0,t) = u(2,t) = 0$

I.C.: $u(x,0) = -3\sin\frac{\pi x}{2} + 23\sin \pi x - 4\sin 2\pi x$

Let $u(x,t) = X(x)T(t)$. Then differentiating and substituting in the PDE yields

$$X''T = 4XT'$$
$$\Rightarrow \frac{X''}{X} = 4\frac{T'}{T}$$

Using the argument that the left hand side is purely a function of x and the right hand side is purely a function of t, and the only way that they can be equal is if they are equal to a constant, we get

$$\frac{X''}{X} = 4\frac{T'}{T} = k \qquad k \text{ a constant}$$

This yields the two *ordinary differential equations*

$$X'' - kX = 0 \qquad \text{and} \qquad T' - \frac{1}{4}kT = 0$$

The boundary condition $u(0,t) = 0$ implies that $X(0)T(t) = 0$. We cannot have $T(t) = 0$, since this would imply that $u(x,t) = 0$. Thus $X(0) = 0$. Similarly, the boundary condition $u(2,t) = 0$ leads to $X(2) = 0$.

We now have the following boundary value problem for $X(x)$:

$$X'' - kX = 0 \qquad X(0) = X(2) = 0$$

This boundary value problem has the solution

$$k = -\left(\frac{n\pi}{2}\right)^2 \qquad X_n(x) = a_n \sin\frac{n\pi}{2}x \qquad n = 1,2,3,\ldots$$

Substituting the values of k into the equation for $T(t)$ leads to

$$T' + \frac{n^2\pi^2}{16}T = 0$$

which has the solution $T_n(t) = c_n e^{-\frac{n^2\pi^2 t}{16}}$, $n = 1,2,3,\ldots$

We now have the solutions

$$u_n(x,t) = A_n \sin\frac{n\pi}{2}xe^{-\frac{n^2\pi^2 t}{16}} \qquad n = 1,2,3,\ldots$$

Since the boundary conditions and the equation are linear and homogeneous, it follows that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{2}xe^{-\frac{n^2\pi^2 t}{16}}$$

satisfies the PDE and the boundary conditions. Since

$$u(x,0) = -3\sin\frac{\pi x}{2} + 23\sin\pi x - 4\sin 2\pi x = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{2}x.$$

Matching the cosine terms on both sides of this equation leads to

$A_1 = -3$ $\qquad A_2 = 23$ and $A_4 = -4$. All of the other constants must be zero, since there are no sine terms on the left to match with them. Thus

$$u(x,t) = -3\sin\frac{\pi x}{2}e^{-\frac{\pi^2}{16}t} + 23\sin\pi xe^{-\frac{\pi^2}{4}t} - 4\sin 2\pi xe^{-\pi^2 t}$$