Ma 530 Power Series II

Please note that there is material on power series at Visual Calculus. Some of this material was used as part of the presentation of the topics that follow.

Operations on Power Series — Addition and Subtraction

Theorem Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ and $g(x) = \sum_{n=0}^{\infty} d_n (x-a)^n$ be power series centered at $x = a$ and let R_f and R_g be their radii of convergence, respectively. Further, let $R = \min(R_f, R_g)$ be the smaller of these two radii. Then the sum and difference of f and g may be computed term by term and the radius of convergence is at least R

Addition:

$$f(x) + g(x) = \sum_{n=0}^{\infty} (c_n + d_n)(x-a)^n$$

Subtraction:

$$f(x) - g(x) = \sum_{n=0}^{\infty} (c_n - d_n)(x-a)^n$$

Example Find the power series expansion for $\frac{5x}{6x^2 - x - 1}$ centered about $x = 0$, and find its radius of convergence.

Solution: We first factor the denominator and do a partial fraction expansion to get

$$\begin{aligned} \frac{5x}{6x^2 - x - 1} &= \frac{5x}{(3x+1)(2x-1)} \\ &= \frac{1}{1+3x} - \frac{1}{1-2x} \quad \text{(Lots of work here)} \end{aligned}$$

Each of these fractions is the sum of a geometric series with a different ratio:

$$\frac{1}{1+3x} = \sum_{n=0}^{\infty} (-3x)^n = \sum_{n=0}^{\infty} (-3)^n x^n \quad \text{for } |3x| < 1 \text{ or } |x| < \frac{1}{3}$$

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n \quad \text{for } |2x| < 1 \text{ or } |x| < \frac{1}{2}$$

Subtracting these, we find

$$\begin{aligned} \frac{5x}{6x^2 - x - 1} &= \sum_{n=0}^{\infty} (-3)^n x^n - \sum_{n=0}^{\infty} 2^n x^n \\ &= \sum_{n=0}^{\infty} [(-3)^n - 2^n] x^n \quad \text{for } |x| < \frac{1}{3} \end{aligned}$$

At this point, we know the radius of convergence is at least $\frac{1}{3}$, but we need to find it explicitly using the ratio test:

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|(-3)^{n+1} - 2^{n+1}\right| |x|^{n+1}}{|(-3)^n - 2^n||x|^n}$$

$$= |x| \lim_{n\to\infty} \frac{\left|(-3)^{n+1} - 2^{n+1}\right|}{|(-3)^n - 2^n|} \cdot \frac{\frac{1}{3^n}}{\frac{1}{3^n}}$$

$$= |x| \lim_{n\to\infty} \frac{\left|\frac{(-3)^{n+1}}{3^n} - \frac{2^{n+1}}{3^n}\right|}{\left|\frac{(-3)^n}{3^n} - \frac{2^n}{3^n}\right|}$$

$$= |x| \lim_{n\to\infty} \frac{\left|(-1)^{n+1}3 - 0\right|}{|(-1)^n - 0|} = 3|x|$$

The series converges when $L = 3|x| < 1$ or $|x| < \frac{1}{3}$. So the the radius of convergence is $R = \frac{1}{3}$ after all.

When is the radius of convergence not the minimum?

Exercise Find the power series expansion for $\frac{4x}{1-4x^2}$ centered about $x = 0$, and find its radius of convergence.. ... ...

Operations on Power Series — Multiplication by Polynomials

Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series centered at $x = a$ and whose radius of convergence is R. Also let k be a constant and let p be a non-negative integer. Then:

Muliplication by a Polynomial:

$$k(x-a)^p f(x) = \sum_{n=0}^{\infty} kc_n (x-a)^{n+p} = \sum_{i=p}^{\infty} kc_{i-p}(x-a)^i$$

which also has radius of convergence R.

The following series was found above, but here is an easier derivation.:

Example Find the power series expansion for $\frac{4x}{1-4x^2}$ centered about $x = 0$, and find its radius of convergence.

Solution: We first use substitution to write $\frac{1}{1-4x^2}$ as a geometric series:

$$\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n = \sum_{n=0}^{\infty} 4^n x^{2n} \quad \text{for } |4x^2| < 1 \text{ or } |x| < \frac{1}{2}$$

Finally we multiply by $4x$:

$$\frac{4x}{1-4x^2} = \sum_{n=0}^{\infty} 4^{n+1} x^{2n+1} \quad \text{for } |x| < \frac{1}{2}$$

Remark This agrees with the previous result

$$\frac{4x}{1-4x^2} = \sum_{k=0}^{\infty} 2^{2k+2} x^{2k+1}$$

Exercise Find the power series expansion for $\frac{3x^2}{1+x^3}$ centered about $x = 0$, and find its radius of convergence... ...

Operations on Power Series — Differentiation

Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series centered at $x = a$ and whose radius of convergence is R.
Then the derivative of f may be computed by differentiating the terms of the series for f:
Differentiation:

$$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1} = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$
$$= \sum_{k=0}^{\infty} (k+1) c_{k+1} (x-a)^k$$

which also has radius of convergence R.

Example Find the power series expansion for $\frac{3x^2}{(1-x^3)^2}$ centered about $x = 0$, and find its radius of convergence.

Solution: We first notice that $\frac{3x^2}{(1-x^3)^2}$ is the derivative of $\frac{1}{1-x^3}$ whose series is

$$\frac{1}{1-x^3} = \sum_{n=0}^{\infty} (x^3)^n = \sum_{n=0}^{\infty} x^{3n} \quad \text{for } |x| < 1$$

So we differentiate this series term by term:

$$\frac{3x^2}{(1-x^3)^2} = \sum_{n=0}^{\infty} 3n x^{3n-1} = \sum_{n=1}^{\infty} 3n x^{3n-1} \quad \text{for } |x| < 1$$

Notice that we can drop the $n = 0$ term because it is zero.

Exercise Find the power series expansion for $\frac{1}{(1+2x)^2}$ centered about $x = 0$, and find its radius of convergence..

Operations on Power Series — Integration

Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series centered at $x = a$ and whose radius of convergence is R.
Then the integral of f may be computed by integrating the terms of the series for f and adding a constant of integration:

Integration:

$$\begin{aligned}\int f(x)\,dx &= \sum_{n=0}^{\infty} c_n \int (x-a)^n\,dx \\ &= \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C \\ &= \sum_{k=1}^{\infty} \frac{c_{k-1}}{k}(x-a)^k + C\end{aligned}$$

which also has radius of convergence R.

Remark If you use a specific antiderivative on the left, then you must evaluate the constant on the right usually by evaluating both sides at the center $x = a$.

Example Find the power series expansion for $\arctan x$ centered about $x = 0$, and find its interval of convergence.

Solution: We first notice that $\arctan x$ is the integral of $\frac{1}{1+x^2}$ whose series is

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} \quad \text{for } |-x^2| < 1 \text{ or } |x| < 1$$

So we integrate this series term by term: (Don't forget the constant!)

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \int x^{2n}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} + C \quad \text{for } |x| < 1$$

To find the constant, we evaluate both sides at $x = 0$.

$$\arctan 0 = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}0^{2n+1} + C$$

Recall $\arctan 0 = 0$ and $0^{2n+1} = 0$. Thus $C = 0$. We substitute back to conclude

$$\arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} \quad \text{for } |x| < 1$$

We check the convergence at the endpoints. At $x = 1$, the series becomes $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ which converges by the Alternating Series Test. At $x = -1$, the series becomes $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(-1)^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}$ which also converges by the Alternating Series Test. So the interval of convergence is $-1 \le x \le 1$.

Example Find a power series representation for

$$f(x) = \frac{1}{(1+x)^3}.$$

We begin with the fact that

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 + \cdots$$

Differentiating, we obtain:

$$-\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 + \cdots$$

and hence

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

Differentiating again, yields

$$\frac{-2}{(1+x)^3} = -2 + 6x - 12x^2 + \cdots$$

Finally, dividing by -2, we obtain:

$$\frac{1}{(1+x)^3} = 1 - 3x + 6x^2 + \cdots$$

Exercise Find the power series expansion for $\ln(1+x)$ centered about $x = 0$, and find its interval of convergence... ...

Remark Mathematicians also use series to define new functions:

Example: The Bessel Function

Example The Bessel function of order 0 is defined by the series

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (n!)^2} x^{2n}$$

Find the domain of $J_0(x)$.

Remark Notice that the series for $J_0(x)$ only has even order terms. This simply means that the coefficients of the odd order terms are all zero.

Solution: We find the radius of convergence by applying the ratio test:

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{x^{2n+2}}{4^{n+1}((n+1)!)^2} \frac{4^n (n!)^2}{x^{2n}}$$

$$= \lim_{n\to\infty} \frac{x^2}{4}\left(\frac{n!}{(n+1)!}\right)^2 = \lim_{n\to\infty} \frac{x^2}{4(n+1)^2} = 0$$

Since $L = 0 < 1$, the series converges for all x and the domain of $J_0(x)$ is $(-\infty, \infty)$.

Remark Since $J_0(x)$ is given by an alternating series, the values of the function may be approximated by taking a finite number of terms and the error will be bounded by the next term.

Taylor and MacLaurin Series

If $f(x)$ has a power series expansion at a, it must be of the form:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This series is called the Taylor Series of the function f at a.

In the special case when $a = 0$, the series is called the MacLaurin Series. In other words,

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

is the MacLaurin Series for $f(x)$.

Remark The *nth* partial sum associated with a Taylor series for a given function is called nth degree Taylor polynomial for $f(x)$ at $x = a$ and is denoted by $T_n(x)$. Thus

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

It will be useful to know the MacLaurin Series for e^x, $\sin x$, and $\cos x$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Example Find the first 5 terms of the Taylor Series for $f(x) = \ln x$ at $a = 2$.

$$f(x) = \ln x \Rightarrow f(2) = \ln 2$$

$$f'(x) = \frac{1}{x} \Rightarrow f'(2) = \frac{1}{2}$$

$$f''(x) = \frac{-1}{x^2} \Rightarrow f''(2) = \frac{-1}{4}$$

$$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(2) = \frac{1}{4}$$

$$f^{(iv)}(x) = \frac{-6}{x^4} \Rightarrow f^{(iv)}(2) = \frac{-6}{16} = \frac{-3}{8}$$

Thus

$$f(x) = \ln x = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{4}\frac{(x-2)^2}{2!} + \frac{1}{4}\frac{(x-2)^3}{3!} - \frac{3}{8}\frac{(x-2)^4}{4!} + \cdots$$

Example Use known MacLaurin Series to find the MacLaurin Series of

$$f(x) = e^{3x}$$

Solution: Since

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$$

$$e^{3x} = 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \cdots$$

Example Use known MacLaurin Series to find the MacLaurin Series of

$$f(x) = \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

Solution:

$$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} + \cdots$$

so

$$\cos 2x = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} + \cdots$$

Then

$$-\cos 2x = -1 + \frac{4x^2}{2!} - \frac{16x^4}{4!} + \cdots$$

and therefore

$$1 - \cos 2x = \frac{4x^2}{2!} - \frac{16x^4}{4!} + \cdots$$

Finally

$$\sin^2 x = \frac{1}{2}[\frac{4x^2}{2!} - \frac{16x^4}{4!} + \cdots]$$

Example Evaluate

$$\int \frac{\sin x}{x}dx$$

Solution:

$$\int \frac{\sin x}{x}dx = \int \frac{1}{x}[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots]dx = \int[1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots]dx = C + x - \frac{x^3}{3(3!)} + \frac{x^5}{5(5!)} + \cdots$$

Example Let α be a nonzero real number. Find the MacLaurin series for

$$f(x) = (1+x)^{\alpha}$$

Solution:

$$\begin{aligned}
f(x) &= (1+x)^{\alpha} \\
f'(x) &= \alpha(1+x)^{\alpha-1} \\
f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} \\
f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} \\
&\vdots \\
f^{(n)}(x) &= \alpha(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-n+1)(1+x)^{\alpha-n}
\end{aligned}$$

Thus

$$f^{(n)}(0) == \alpha(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-n+1)$$

So the MacLaurin series is given by

$$\begin{aligned}
(1+x)^{\alpha} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-n+1)}{n!}x^n \\
&= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots
\end{aligned}$$

Using the Ratio Test we have that

$$\begin{aligned}
L &= \lim_{n\to\infty}\left| \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-n)}{(n+1)!}x^{n+1} \bullet \frac{n!}{\alpha(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-n+1)x^n} \right| \\
&= \lim_{n\to\infty}\left| \frac{(\alpha-n)x}{n+1} \right| = |x|
\end{aligned}$$

Thus this series converges for $|x| < 1$.