Ma 530

Linear Differential Equations

We shall now begin a detailed study of the second-order linear differential equation

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$

Fundamental theory of second-order linear equations

The following theorem gives information concerning the existence of solutions of second-order linear differential equations. We shall accept it as valid without proof.

Theorem 1: Consider the Initial Value Problem

$$\text{D.E. } a(x)y'' + b(x)y' + c(x)y = f(x)$$

$$\text{I.C. } y(x_0) = y_0 \qquad y'(x_0) = y_0'$$

If $a(x),\ b(x),\ c(x),\ f(x)$ are all continuous functions in the interval I, where $x_0 \in I$ and $a(x) \neq 0$ for all x in I, then the IVP possesses a unique solution. This solution has a continuous derivative and is defined throughout I.

Example

$$D.E.\ a(x)y'' + b(x)y' + c(x)y = 0 \qquad \text{Homogeneous Equation}$$

$$\text{I.C. } y(x_0) = 0 \qquad y'(x_0) = 0$$

One solution is $y(x) \equiv 0$. Theorem 1 $\Rightarrow$ only solution is $y \equiv 0$.
We shall assume from now on that $a, b, c,$ and f are continuous in a common interval I and $a(x) \neq 0$ in I so that Theorem 1 holds.

Notation: Let

$$L[y] \equiv a(x)y'' + b(x)y' + c(x)y.$$

Then $L[2] = 2c(x)$

$L[3x] = 3b(x) + 3xc(x).$

With this notation the second order differential equation $a(x)y'' + b(x)y' + c(y)y = f(x)$ can be written as $L[y]= f(x)$. The homogeneous case is when $f(x) = 0 \Rightarrow L[y] = 0$. This is called the homogeneous equation. If $f(x) \neq 0 \Rightarrow$ a nonhomogeneous equation.
$L[y]$ is called a <u>linear</u> <u>operator</u> because it has the following property.

Theorem 2:

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2]$$

where y_1 and y_2 are any twice differential functions and c_1 and c_2 are any constants.
Proof:

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= a(x)\ (c_1y_1 + c_2y_2)'' + b(x)\ (c_1y_1 + c_2y_2)' + c(x)\ (c_1y_1 + c_2y_2) \\ &= a(x)\ (c_1y_1'' + c_2y_2'') + b(x)\ (c_1y_1' + c_2y_2')\ + c(x)\ (c_1y_1 + c_2y_2) \\ &= c_1[a(x)y_1'' + b(x)y_1' + c(x)y_1] + c_2[a(x)y_2'' + b(x)y_2' + c(x)y_2] \end{aligned}$$

$\Rightarrow L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2]$

Properties of solutions of second order equations.

Theorem 3: If $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation $L[y] = 0$, then $y = c_1y_1(x) + c_2y_2(x)$ is also a solution.
Proof. $L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2]$ from above. Since y_1 is a solution of $L[y] = 0 \Rightarrow L[y_1] = 0$. Similarly $L[y_2] = 0$.
Hence $L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2] = 0 + 0 = 0 \Rightarrow y = c_1y_1 + c_2y_2$ is also a solution of $L(y) = 0$.

Example $y'' - 9y = 0$ e^{3x} and e^{-3x} are solutions. Theorem 3 $\Rightarrow y = c_1e^{3x} + c_2e^{-3x}$ is also a solution.

Remark. We desire to be able to find the general solution of $L[y] = 0$. The above theorem tells us that if y_1 and y_2 are solutions, then $c_1y_1 + c_2y_2$ is a solution, but it does not tell us that this is the general solution. In order to know when one has a general solution it is necessary to introduce the concept of the linear independence of two functions.

Definition: Two functions $y_1(x)$ and $y_2(x)$ are called linearly dependent (LD) in an interval I if it is possible to find two constants c_1 and c_2, not both zero, so that

$$c_1y_1(x) + c_2y_2(x) = 0 \ \ \forall x \in I.$$

Two functions are called linearly independent (LI) if they are not linearly dependent, i.e., if

$$c_1y_1(x) + c_2y_2(x) = 0 \ \ \forall x \in I \Rightarrow c_1 = c_2 = 0.$$

Remark. If two functions are LD in $I \Rightarrow$ one of the functions is equal to a constant times the other in I.

Example (a) $x,\ 2x$ are LD in any interval I, since

$$(-2)x + (1)\ 2x = 0 \quad \forall x \in I$$

(b) $x^2,\ x$ are LI in any interval I, since

$$c_1x^2 + c_2x = 0 \quad \forall x \in I$$

is impossible because this equation has at most two real roots in I. Thus, we must have $c_1 = c_2 = 0$.

(c) Two functions are *LD* if one of them is the zero function. If $y_1 \equiv 0$, then

$$c_1 y_1 + 0 \cdot y_2 = c_1 0 + 0 \cdot y \equiv 0 \ \forall x \in I$$

and any $c_1 \neq 0$.

(d) If $\lambda_1 \neq \lambda_2$, then $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are LI for if

$$c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \equiv 0$$

$\Rightarrow$

$$c_1 \equiv -c_2 e^{(\lambda_2 - \lambda_1)x}.$$

But c_1 is a constant and therefore the last equation $\Rightarrow \lambda_1 = \lambda_2$, which is a contradiction.

Facts from algebra needed in the proofs of the next theorems.

1. $\left.\begin{array}{l} d_1 x + d_2 y = d_3 \\ e_1 x + e_2 y = e_3 \end{array}\right\}$ has a unique solution $\Leftrightarrow \begin{vmatrix} d_1 & d_2 \\ e_1 & e_2 \end{vmatrix} \neq 0$

   If $d_3 = e_3 = 0$ and $det \neq 0 \Rightarrow x = y = 0$ is the only solution.

2. $\left.\begin{array}{l} d_1 x + d_2 y = 0 \\ e_1 x + e_2 y = 0 \end{array}\right\}$ has nontrivial solution. $\Leftrightarrow \begin{vmatrix} d_1 & d_2 \\ e_1 & e_2 \end{vmatrix} = 0$

Definition: The Wronskian of two differentiable functions y_1 and y_2 is defined to be

$$W[y_1(x),\ y_2(x)] \equiv \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \equiv y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

Theorem 4. If $W[y_1(x), y_2(x)]$ is different from zero for at least one point in an interval I, then $y_1(x)$ and $y_2(x)$ are *LI* in I.

Proof. Suppose $y_1,\ y_2$ are *LD*. Then $\exists$ constants c_1, c_2, not both zero, such that

$$\left.\begin{array}{l} c_1 y_1(x) + c_2 y_2(x) \equiv 0 \\ c_1 y_1'(x) + c_2 y_2'(x) \equiv 0 \end{array}\right\}$$

By assumption these two equations have a nontrivial solution c_1, c_2 at each point x in I. Therefore the determinant of the coefficients (by 2) must be zero for each x. But the determinant of coefficients $= W[y_1(x), y_2(x)]$ and $W \neq 0$ for at least one point in I. $\Rightarrow y_1$ and y_2 are not *LD*.

Corollary. If y_1, y_2 are *LD* in $I \Rightarrow W[y_1(x),\ y_2(x)] \equiv 0$ in I.

Remark. Converse of Theorem 4 is not true in general, i.e., there exist functions which are *LI* in an interval I and whose Wronskian is $\equiv 0$ in I.

However, if y_1 and y_2 are solutions of $L[y] = 0$ then the following converse holds.

Theorem 5. If $y_1(x),\ y_2(x)$ are *LI* solutions of $L[y] = 0$ in I, then $W[y_1(x),\ y_2(x)]$ is never zero in I.

Proof. If $W[y_1(x), y_2(x)] = 0$ for some $x_0 \in I$, then the equations

$$c_1 y_1(x_0) + c_2 y_2(x_0) = 0$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0$$

have a nontrivial solution, i.e. $\exists\ c_1, c_2$ not both zero satisfying the system. For these values of c_1 and c_2 the function $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is a solution of $L[y] = 0$ and satisfies the initial conditions $y(x_0) = 0, y'(x_0) = 0$. However, by Theorem 1 the only solution of this problem is $y(x) \equiv 0 \Rightarrow c_1 y_1(x) + c_2 y_2(x) \equiv 0\ \forall x \in I \Rightarrow y_1, y_2$ are *LD*. Contradiction! $\Rightarrow W[\ ]$ is never zero in I.

Corollary. The Wronskian of 2 solutions of $L[y] = 0$ is either identically zero (if solutions are *LD*) or never zero (if solutions are *LI*).

Theorem 6. If $y_1(x)$ and $y_2(x)$ are *LI* solutions of $L[y] = 0$, then $y = c_1 y_1 + c_2 y_2$ is the general solution of $L[y] = 0$.

Example e^{3x} and e^{-3x} are *LI* solutions of $y'' - 9y = 0 \Rightarrow$ general solution is $y = c_1 e^{3x} + c_2 e^{-3x}$.

Theorem 6 tells us that the problem of finding the general solution of $L[y] = 0$ is reduced to finding any two linearly independent solutions.

Example This example is a video slide show. Slide Example

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Question: Do two *LI* solutions of $L[y] = 0$ actually exist? The answer is given in the affirmative by the next theorem.

Theorem 7. $\exists$ two linear independent solutions of $L[y] = 0$.

Proof. Let $y_1(x)$ be the unique solution of $L[y] = 0$ with initial conditions $y_1(x_0) = 1, y_1'(x_0) = 0$, and $y_2(x)$ be the unique solution of $L[y] = 0$ with initial conditions $y_2(x_0) = 0, y_2'(x_0) = 1$. Note that y_1 and y_2 exist by Theorem 1. Now y_1 and y_2 are *LI* by Theorem 5 since

$$W[y_1(x_0), y_2(x_0)] = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Theorem 8. If y_p is any particular solution of the nonhomogeneous equation $L[y] = f(x)$ and y_h is the general solution of the homogeneous equation $L[y] = 0$, then the general solution of $L[y] = f(x)$ is $y = y_p + y_h$.

Example Solve $y'' - 9y = e^x$
We know that $y_h = c_1e^{3x} + c_2e^{-3x}$. $y_p = ?$ Assume $y_p = Ae^x$

$\Rightarrow Ae^x - 9Ae^x = e^x \Rightarrow -8A = 1 \qquad A = -\frac{1}{8} \Rightarrow y_p = -\frac{1}{8}e^x$
$\Rightarrow y = c_1e^{3x} + c_2e^{-3x} - \frac{1}{8}e^x$ is the general solution.

Theorem 9. Principle of superposition. If y_1 is a solution of $L[y] = f_1$ and y_2 is a solution of $L[y] = f_2$, then $y = y_1 + y_2$ is a solution of $L[y] = f_1 + f_2$.

Example Solve $y'' - 9y = e^x + 5$. Before we found that $y = -\frac{1}{8}e^x$ was a particular solution of $y'' - 9y = e^x$. To find
a particular solution of $y'' - 9y = 5$ assume $y \equiv k \Rightarrow k = -\frac{5}{9}$. The general solution of equation is therefore

$$y = c_1e^{3x} + c_2e^{-3x} - \frac{1}{8}e^x - \frac{5}{9}.$$

Extension: If for $i = 1, 2, \ldots, n$, y_i is a solution of $L[y] = f_i$, then $\sum_{i=1}^{n} y_i$ is a solution of $L[y] = \sum_{i=1}^{n} f_i$.

Complex-valued Solutions

A complex-valued function f of a real variable x is a function of the form

$$f(x) = u(x) + iv(x)$$

where $u(x)$ and $v(x)$ are real functions and $i = \sqrt{-1}$.

Definition. If $f = u + iv$, u, v real functions, then f is continuous if u and v are continuous; f is differential if u and v are differential and

$$f'(x) = u'(x) + iv'(x).$$

Example a) $f(x) = 3x + ix^2 \Rightarrow f'(x) = 3 + 2ix$

b) $\frac{d}{dx}(3x + ix^2)^2 = 2(3x + ix^2)(3 + 2ix) = 2(9x - 2x^3 + 9ix^2)$

c) Let

$$E(x) = e^{ax}(\cos bx + i\sin bx)$$

Then

$$\begin{aligned} E'(x) &= ae^{ax}(\cos bx + i\sin bx) + e^{ax}(-b\sin bx + bi\cos bx) \\ &= e^{ax}[a(\cos bx + i\sin bx) + bi(\cos bx + i\sin bx)] \\ &= e^{ax}[a + bi](\cos bx + i\sin bx). \end{aligned}$$

Hence

$$E'(x) = (a+bi)E(x).$$

Based on this we define the complex exponential via

$$e^{(a+bi)x} = e^{ax}\cos bx + ie^{ax}\sin bx$$

$a = 0 \Rightarrow$

$$e^{bix} = \cos bx + i\sin bx.$$

This is called Euler's formula. Hence

$$e^{(a+bi)x} = e^{ax} \bullet e^{bix}.$$

Example $y = e^{ix}$ satisfies $y'' + y = 0$ since $y' = ie^{ix}$ $y'' = -e^{ix} \Rightarrow -e^{ix} + e^{ix} = 0.$

The theorem below gives the connection between real and complex solutions of a linear differential equation with real coefficients.

Theorem 1. Consider the differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

where $a(x)$, $b(x)$, and $c(x)$ are real functions. The complex function $y = u + iv$, where u and v are real, is a solution of this equation $\Leftrightarrow$ u and v are solutions.

Proof. As before we denote the equation by $L[y] = 0$. It is easily shown that $L[y] = L[u] + iL[v]$ where $L[u]$ and $L[v]$ are real. Therefore y is a solution $\Leftrightarrow L[y] = L[u] + iL[v] \equiv 0$. Since a complex number is zero $\Leftrightarrow$ its real and imaginary parts are zero,
$\Rightarrow L[y] = 0 \Leftrightarrow L[u] = 0$ and $L[v] = 0 \Leftrightarrow u$ and v solutions.

Example $y = e^{ix}$ is a solution of $y'' + y = 0$. Since $e^{ix} = cosx + isinx \Rightarrow cosx$ and $sinx$ are solutions. This is easily verified.

Homogeneous Linear Equations with Constant Coefficients

We shall now discuss the problem of solving the homogeneous equation

$$ay'' + by' + cy = 0 \ (*)$$

where a, b and c are real constants and $a \neq 0$.

Possible candidates for a solution are x and powers of x. These are no good. $\ln x$ is also no good. We shall try $e^{\lambda x}$. If $y = e^{\lambda x}$ is a solution of (*)
$\Rightarrow a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = e^{\lambda x}(a\lambda^2 + b\lambda + c) = 0$. This is to be a solution $\forall x$. $\Rightarrow$

$$a\lambda^2 + b\lambda + c = 0.$$

This equation for λ is called the *auxiliary* or *characteristic* equation.

It has the solution $\lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ $\quad \Delta = b^2 - 4\,ac$

There are three possibilities:

(1) $\Delta > 0$ two real, distinct roots

(2) $\Delta = 0$ one real root, repeated

(3) $\Delta < 0$ two imaginary roots which are the complex conjugates of each other, i.e. if $\lambda_1 = \alpha + i\beta \Rightarrow \lambda_2 = \alpha - i\beta$

We shall now discuss the three cases in detail.

Case 1. $\Delta > 0$. There are two real distinct roots λ_1, λ_2, where $\lambda_1 \neq \lambda_2$

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$\Rightarrow e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are both solutions of the differential equation. These functions are LI, $\Rightarrow$ general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

where λ_1 and λ_2 are both real and $\lambda_1 \neq \lambda_2$.

Example $2y'' - y' - 3y = 0$

$\Rightarrow 2\lambda^2 - \lambda - 3 = 0$

or

$(2\lambda - 3)(\lambda + 1) = 0 \Rightarrow \lambda_1 = -1 \;\; \lambda_2 = \frac{3}{2} \;\; \Rightarrow y = c_1 e^{-x} + c_2 e^{+\frac{3}{2}x}$.

Case 2. $\Delta = 0$. There is one real, repeated root $\lambda_1 = -\dfrac{b}{2a} \Rightarrow e^{\lambda_1 x}$ is a solution. We need a second LI solution. To find it we shall use the method of variation of parameters. We seek a solution of the form

$$y = v(x) e^{\lambda_1 x},$$

where $v(x)$ is a function to be determined. Now

$$y' = v' e^{\lambda_1 x} + v\lambda_1 e^{\lambda_1 x}$$

and

$$y'' = v'' e^{\lambda_1 x} + 2v'\lambda_1 e^{\lambda_1 x} + v\lambda_1^2 e^{\lambda_1 x}$$

$\Rightarrow$

$$av''(x) e^{\lambda_1 x} + 2av'\lambda_1 e^{\lambda_1 x} + av\lambda_1^2 e^{\lambda_1 x} + bv' e^{\lambda_1 x} + bv\lambda_1 e^{\lambda_1 x} + cve^{\lambda_1 x} = 0$$

$\Rightarrow$

$$av'' + \underbrace{(2a\lambda_1 + b)}\,v' + \underbrace{(a\lambda_1^2 + b\lambda_1 + c)}\,v = 0$$

$\Rightarrow v'' = 0$ (why?) $\Rightarrow$

$$v = c_1 + c_2 x$$

⇒

$$y = ve^{\lambda_1 x} = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$$

is a solution of the differential equation in the case where there exists one repeated root λ_1. Since $e^{\lambda_1 x}$ and $xc^{\lambda_1 x}$ are LI ⇒ this is the general solution.

Example $y'' - 4y' + 4y = 0$
$\lambda^2 - 4\lambda + 4 = 0$ or $(\lambda - 2)^2 = 0$ ⇒ one real, repeated root $\lambda = 2$. ⇒

$$y = c_1 e^{2x} + c_2\, x e^{2x}.$$

Case 3. $\Delta < 0$ 2 complex roots

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{i\sqrt{4ac - b^2}}{2a}$$

⇒

$$\lambda_1 = \alpha + i\beta = -\frac{b}{2a} + i\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}$$
$$\lambda_2 = \alpha - i\beta = -\frac{b}{2a} - i\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}}$$

where α and β are real numbers.
⇒ two complex solutions. $e^{(\alpha + i\beta)x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$ and $e^{(\alpha - i\beta)x} = e^{\alpha x}(\cos\beta x - i\sin\beta x)$

Since the differential equation has real coefficients, ⇒ real and imaginary parts of above are solutions, i.e.,
$e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are both solutions in this case. These are LI functions.
⇒ the solution is

$$y = e^{\alpha x}(A cos\beta x + B sin\beta x) \quad \text{where } A, B \quad \text{real constants.}$$

Example

$$32y'' - 40y' + 17y = 0$$

$$32\lambda^2 - 40\lambda + 17 = 0$$

$$\lambda = \frac{40 \pm \sqrt{1600 - 4(32)(17)}}{2(32)} = \frac{40 \pm \sqrt{1600 - 2176}}{2(32)} = \frac{40 \pm i\sqrt{576}}{2(32)} = \frac{5 \pm \sqrt{9}}{8} = \frac{5}{8} \pm \frac{3}{8}\, i$$

Thus

$$\lambda_1 = \frac{5}{8} + \frac{3}{8}i \text{ and } \lambda_2 = \frac{5}{8} - \frac{3}{8}i$$

⇒

$$y = e^{\frac{5}{8}x}(A\cos\frac{3}{8}x + B\sin\frac{3}{8}x)$$

Example Write down a second order homogeneous linear differential equation with real constant coefficients whose solutions are

$$\frac{1}{2}e^{-2x}\cos 3x \text{ and } \frac{3e^{-2x}}{4}\sin 3x.$$

$\Rightarrow \alpha = -2 \quad \beta = 3$ so that $\lambda_1 = -2 + 3i$ and $\lambda_2 = -2 - 3i$.

$\Rightarrow$

$$\begin{aligned} p(\lambda) &= [\lambda - (-2 + 3i)][\lambda - (-2 - 3i)] \\ &= [\lambda + 2 - 3i][\lambda + 2 + 3i] \\ &= \lambda^2 + (2 + 3i)\lambda + (2 - 3i)\lambda + 4 + 9 \\ &= \lambda^2 + 4\lambda + 13 \end{aligned}$$

(Check: $\lambda = \dfrac{-4 \pm \sqrt{16 - 4(1)(13)}}{2} = \dfrac{-4 \pm bi}{2} = -2 \pm 3i$)

$\Rightarrow$ equation is

$$y'' + 4y' + 13y = 0$$

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Undetermined Coefficients

Let us now consider the problem of solving

$$ay'' + by' + cy = f(x) \quad a \neq 0 \quad (*)$$

a, b, c real constants. We know that the general solution is $y = y_h + y_p$, where

$$y_h = \text{the solution of the homogeneous equation}$$

and

$$y_p = \text{a particular solution of the equation}$$

We know how to find y_h . We shall now discuss ways of finding y_p for certain special functions $f(x)$.

1. $f(x) = Ke^{\alpha x}$ K constant, α constant.

Thus we seek y_p for

$$ay'' + by' + cy = Ke^{\alpha x}.$$

Due to the exponential form of $f(x)$ we seek y_p of the form

$$y_p = Ae^{\alpha x}$$

A = ? The differential equation (*) ⇒

$$(a\alpha^2 + b\alpha + c)Ae^{\alpha x} = Ke^{\alpha x}$$

⇒

$$A = \frac{K}{a\alpha^2 + b\alpha + c}$$

⇒

$$y_p = \frac{Ke^{\alpha x}}{a\alpha^2 + b\alpha + c}$$

The above is a particular solution provided the denominator is non-zero. Note that the denominator is $p(\lambda) = a\lambda^2 + b\lambda + c$ with $\lambda = \alpha$. This is the characteristic polynomial with $\lambda = \alpha$.

If $p(\alpha) = 0$, ⇒ we do not have a y_p . However, $p(\alpha) = 0 \Rightarrow \alpha$ is a root of characteristic equation. ⇒ $e^{\alpha x}$ is solution of the homogeneous equation, and therefore $Ae^{\alpha x}$ cannot be a solution of the nonhomogeneous equation. If $p(\alpha) = 0$, we try

$$y_p = Axe^{\alpha x}$$

⇒ $y_p' = A\alpha xe^{\alpha x} + Ae^{\alpha x}$ and $y_p'' = A\alpha^2 xe^{\alpha x} + A\alpha e^{\alpha x} + A\alpha e^{\alpha x} = A\alpha^2 xe^{\alpha x} + 2A\alpha e^{\alpha x}$
Substitution into the differential equation $(*)$ ⇒

$$Axe^{\alpha x}[a\alpha^2 + b\alpha + c] + Ae^{\alpha x}[2a\alpha + b] = Ke^{\alpha x}$$

⇒

$$A = \frac{K}{2a\alpha + b}$$

⇒

$$y_p = \frac{Kxe^{\alpha x}}{2a\alpha + b} \quad \text{if} \quad p(\alpha) = 0$$

provided, of course, that $2a\alpha + b \neq 0$. Note that $p(\lambda) = a\lambda^2 + b\lambda + c \Rightarrow p'(\lambda) = 2a\lambda + b$
⇒

$$y_p = \frac{Kxe^{\alpha x}}{p'(\alpha)} \quad \text{when } p(\alpha) = 0 \text{ and } p'(\alpha) \neq 0$$

If $p(\alpha) = 0$ and $p'(\alpha) = 0$ ⇒ above y_p is no good. But $p'(\alpha) = 0 \Rightarrow 2a\alpha + b = 0 \Rightarrow \alpha = -\frac{b}{2a}$. ⇒ α is a double (repeated) root of $a\lambda^2 + b\lambda + c = 0$. Hence both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of the homogeneous equation if $p(\alpha) = p'(\alpha) = 0$, and these cannot therefore be solutions of the nonhomogeneous equation. If $p(\alpha) = p'(\alpha) = 0$ we try

$$y_p = Ax^2e^{\alpha x}.$$

Differentiating and substituting into the equation leads to
⇒

$$A = \frac{K}{2a} = \frac{K}{p''(\alpha)}$$

since $p'(\lambda) = 2a\lambda + b \quad \Rightarrow \quad p''(\lambda) = 2a$
Thus

$$y_p = \frac{K}{p''(\alpha)} x^2 e^{\alpha x} \quad \text{if } p(\alpha) = p'(\alpha) = 0.$$

$p''(\alpha) \neq 0$ since $a \neq 0$ by assumption.

Summary: A particular solution of $L[y] = ke^{\alpha x}$ is

$$y_p = \frac{Ke^{\alpha x}}{p(\alpha)} \quad \text{if } p(\alpha) \neq 0$$

$$y_p = \frac{Kxe^{\alpha x}}{p'(\alpha)} \quad \text{if } p(\alpha) = 0, p'(\alpha) \neq 0$$

$$y_p = \frac{K}{p''(\alpha)} x^2 e^{\alpha x} \quad \text{if } p(\alpha) = p'(\alpha) = 0$$

Example (a) $y'' - 5y' + 4y = 2e^x$

Homogeneous solution: $p(\lambda) = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1) \quad \Rightarrow \lambda = 4,\ 1 \Rightarrow y_h = c_1 e^x + c_2 e^{4x}$

Now to find a particular solution for $2e^x$. $\Rightarrow \alpha = 1 \quad p(1) = 0$ Since $p'(\lambda) = 2\lambda - 5$

$p'(1) = 2 - 5 = -3 \neq 0$

$\Rightarrow$

$$y_p = \frac{kxe^{\alpha x}}{p'(\alpha)} = \frac{2xe^x}{-3}$$

$\Rightarrow$

$$y = y_h + y_p = c_1 e^x + c_2 e^{4x} - \frac{2}{3} xe^x$$

(b) $y'' - 5y' + 4y = 3 + 2e^x$

$y_h = c_1 e^x + c_2 e^{4x} \quad p(\lambda) = \lambda^2 - 5\lambda + 4$

Consider $y'' - 5y' + 4y = 3$

$3 = ke^{\alpha x}$ with $k = 3 \quad \alpha = 0 \quad p(0) = 4 \neq 0 \quad \Rightarrow y_p' = \frac{3}{4}$

$\Rightarrow$

$$y = c_1 e^x + c_2 e^{4x} + \frac{3}{4} - \frac{2}{3} xe^x$$

2. $f(x) = k \cos \beta x$ or $f(x) = k \sin \beta x$

For example, we seek a particular solution of

$$L[y] = ay'' + by' + cy = k cos \beta x$$

We shall use the complex exponential to solve for y_p. Recall

$$ke^{i\beta x} = k \cos \beta x + ik \sin \beta x.$$

Hence we consider also the equation

$$L[v] = av'' + bv' + cv = k \sin \beta x$$

By multiplying this last equation by i and adding the result to $L[y] \Rightarrow$

$$L[y] + iL[v] = k \cos \beta x + ik \sin \beta x = ke^{i\beta x}.$$

But $iL[v] = L[iv]$, since L is linear. Hence if we let $w = y + iv \Rightarrow$ the equation

$$aw'' + bw' + cw = ke^{i\beta x}$$

or

$$L[y] + iL[v] = L[y] + L[iv] = L[y + iv] = L[w] = ke^{i\beta x}$$

and therefore we have the complex equation $L[w] = ke^{i\beta x}$ for w. To find w_p for this we use the formulas derived above. Then we find y_p from $y_p = \text{Re}\, w_p =$ real part of w_p. For $f(x) = k\sin\beta x$ we have $y_p = \text{Im}\, w_p =$imaginary part of w_p.

Example Find a particular solution of

$$y'' + 7y' + 12y = 3cos2x$$

Let $w = y + iv \Rightarrow$ find w_p for $w'' + 7w' + 12w = 3e^{2ix}$. Now $p(\lambda) = \lambda^2 + 7\lambda + 12 \Rightarrow$
$p(\alpha) = p(2i) = (2i)^2 + 7(2i) + 12 = -4 + 14i + 12 \neq 0$
$\Rightarrow$

$$w_p = \frac{3e^{2ix}}{p(2i)} = \frac{3e^{2ix}}{8 + 14i}.$$

$y_p = \text{Re}\, w_p = ?$
To find y_p we shall rationalize the denominator.

$$\begin{aligned} w_p &= \frac{3e^{2ix}}{8 + 14i} \times \frac{8 - 14i}{8 - 14i} \\ &= \frac{3(8 - 14i)e^{2ix}}{64 + 196} \\ &= \frac{3(8 - 14i)e^{2ix}}{260} \\ &= \frac{3}{260}(8 - 14i)[\cos 2x + i\sin 2x] \\ &= \frac{3}{260}[8\cos 2x + 14\sin 2x] + \frac{3}{260}i[8\sin 2x - 14\cos 2x] \end{aligned}$$

Thus

$$y_p = \text{Re}\, w_p = \frac{3}{260}[8\cos 2x + 14\sin 2x]$$

Example

$$y'' + 4y = 3\sin 2x$$

$\Rightarrow$

$$w'' + 4w = 3e^{2ix}$$

$p(\lambda) = \lambda^2 + 4 \Rightarrow p(2i) = 0$ and $p'(\lambda) = 2\lambda$. Now $p'(2i) \neq 0$
$\Rightarrow$

$$\begin{aligned} w_p &= \frac{3xe^{2ix}}{p'(2i)} = \frac{3xe^{2ix}}{4i} \times \frac{i}{i} = -\frac{3}{4}ixe^{2ix} = -\frac{3}{4}ix[\cos 2x + i\sin 2x] \\ &= -\frac{3}{4}xi\cos 2x + \frac{3}{4}x\sin 2x \end{aligned}$$

$\Rightarrow$

$$y_p = \text{Im}\, w_p = -\frac{3}{4} x \cos 2x$$

Example $y'' + 7y' + 12y = 3\cos 2x$ again.
Let $y_p = A\cos 2x + B\sin 2x$
$y_p' = -2A\sin 2x + 2B\cos 2x \qquad y_p'' = 4A\cos 2x - 4B\sin 2x$
$\Rightarrow$

$$-4A\cos 2x - 4B\sin 2x - 14A\sin 2x + 14B\cos 2x + 12A\cos 2x + 12B\sin 2x = 3\cos 2x$$

$\Rightarrow$

$$\cos 2x[8A + 14B] + \sin 2x[8B - 14A] = 3\cos 2x$$

$\Rightarrow$

$$8A + 14B = 3 \quad 8B - 14A = 0 \;\Rightarrow\; B = \frac{7}{4}A$$

$8A + \frac{7}{2}(7)A = 3 \quad 8A + \frac{49}{2}A = 3 \;\Rightarrow\; \frac{16+49}{2}A = 3 \quad A = \frac{6}{65} \;\Rightarrow B = \frac{21}{130}$
$\Rightarrow\; y_p = \frac{6}{65}\cos 2x + \frac{21}{130}\sin 2x$ as before.

III. $f(x) = B_0 + B_1x + \cdots + B_nx^n$ polynomial.

We want y_p for

$$ay'' + by' + cy = B_0 + B_1x + \cdots + B_nx^n$$

We try a solution of the form

$$y_p = Q_n(x) = A_0 + A_1x + \cdots + A_nx^n$$

If $p(0) \neq 0$, then when we substitute Q_n into the equation we will get a polynomial of degree n and we can determine $A_k'\,s$ by equating coefficients of like powers of x. If $p(0) = 0$, but $p'(0) \neq 0$ use $y_p = xQ_n(x)$. Similarly if $p(0) = p'(0) = 0$ take $y_p = x^2Q_n(x)$.

Example

$$y'' + 3y' = 2x^2 + 3x$$

In this example the right hand side is a polynomial of degree 2.
$p(\lambda) = \lambda^2 + 3\lambda$ so $p(0) = 0$. $\quad p'(\lambda) = 2\lambda + 3$ and $p'(0) \neq 0$
$\Rightarrow$

$$y_p = xQ_2(x) = x(A_0 + A_1x + A_2x^2) = A_0x + A_1x^2 + A_2x^3$$

$\Rightarrow$

$$y_p' = A_0 + 2A_1x + 3A_2x^3 \;\Rightarrow\; y_p'' = 2A_1 + 6A_2x$$

The differential equation $\Rightarrow$

$$2A_1 + 6_2x + 3A_0 + 6A_1x + 9A_2x^2 = 2x^2 + 3x$$

$\Rightarrow$

$$2A_1 + 3A_0 = 0 \text{ and } 6A_2 + 6A_1 = 3 \text{and } 9A_2 = 2.$$

$\Rightarrow A_2 = \frac{2}{9} \quad A_2 + A_1 = \frac{1}{2} \quad \frac{2}{9} + A_1 = \frac{1}{2} \Rightarrow A_1 = \frac{1}{2} - \frac{2}{9} = \frac{9-4}{18} = \frac{5}{18}$
$2\,(\frac{5}{18}) + 3A_0 = 0 \quad A_0 = -\frac{10}{18(3)} = -\frac{5}{27} \Rightarrow$

$$y_p = -\frac{5}{27}x + \frac{5}{18}x^2 + \frac{2}{9}x^3$$

IV. $\boldsymbol{f(x) = (B_0 + B_1 x + \cdots + B_n x^n) e^{\alpha x}}$

We want a particular solution for the DE

$$ay'' + by' + cy = (B_0 + B_1 x + \cdots + B_n x^n) e^{\alpha x}$$

We seek a solution of the form

$$y_p = Q_n(x) e^{\alpha x} \text{ if } p(\alpha) \neq 0$$
$$y_p = x Q_n(x)^{\alpha x} \text{ if } p(\alpha) = 0, \text{ and } p'(\alpha) \neq 0$$
$$y_p = x^2 Q_n(x) e^{\alpha x} \text{ if } p(\alpha) = p'(\alpha) = 0$$

Example These examples are video slide shows. Slide Example 1 Slide Example 2

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Variation of Parameters

Let us now consider the non-homogeneous equation

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$

where a, b, c, f are continuous *functions* in some internal I and $a(x) \neq 0\ \forall x \in I$. Note we are not assuming that a, b, and c are constants. We seek y_p, a particular solution. We shall use the method of variation of parameters.
If $y_1(x)$ and $y_2(x)$ are two (known) LI solutions of the homogeneous equation ⇒

$$y_h = c_1\ y_1(x) + c_2\ y_2(x).$$

To find y_p we shall replace c_1 and c_2 by unknown functions of x and seek to determine these functions. Hence let

$$y_p = v_2(x) y_1(x) + v_2(x) y_2(x)$$

Substitution of the above into the differential equation ⇒ only one condition for v_1 and v_2. We may therefore impose another condition arbitrarily but in such a manner as to simplify things.
Now

$$y_p' = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2$$

If we require

$$v_1'\ y_1 + v_2'\ y_2 \equiv 0 \quad (*)$$

then no second derivatives of v_1 and v_2 will appear in y_p''. We therefore make this one condition. The other comes from the differential equation. Now $(*)$ ⇒

$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2 y_2' + v_2 y_2''$$

Substituting into the differential equation implies

$$v_1(\underbrace{ay_1'' + by_1' + cy_1}) + v_2(\underbrace{ay_2'' + by_2' + cy_2}) + av_1'y_1' + av_2'y_2' = f(x)$$

Since the "lower bracketed" quantities are zero, this last equation ⇒

$$v_1'y_1' + v_2'y_2' = \frac{f(x)}{a(x)}$$

.

This is a second condition for v_1' and v_2'.

⇒ we have found two equations to determine v_1, v_2 namely

$$v_1'y_1 + v_2'y_2 \equiv 0$$

and

$$v_1'y_1' + v_2'y_2' \equiv \frac{f(x)}{a(x)}$$

These two equations can be solved for v_1', v_2' provided

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

However, the above is the Wronskian of y_1 , y_2 and is never zero since y_1 and y_2 are LI. Hence

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{f}{a} & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

and

$$v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{f}{a} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

⇒

$$v_1 = -\int \frac{y_2 f(x)}{a(x)\ W[y_1, y_2]} dx$$

and

$$v_2 = \int \frac{y_1 f(x)}{a(x)\ W[y_1, y_2]} dx$$

The particular solution to non-homogeneous equation is

$$y_p = v_1(x)\ y_1(x) + v_2(x)\ y_2(x)$$

with v_1 and v_2 given by the above expressions.

Example .

$$y'' + y = \sec x$$

$\Rightarrow$ let $y_1 = \cos x$ and $y_2 = \sin x$, since these are the two LI homogeneous solutions. Then we take

$$y_p = v_1(x)\cos x + v_2(x)\sin x$$

The two conditions given above $\Rightarrow$

$$v_1'\cos x + v_2'\sin x = 0$$
$$-v_1'\sin x + v_2'\cos x = \sec x.$$

Since $W[y_1, y_2] = \cos^2 x + \sin^2 x = 1$

$$v_1 = -\int \frac{\sin x\ \sec x}{1}dx = -\int \frac{\sin x}{\cos x}dx = \ln|\cos x|$$

and

$$v_2 = \int \cos x \sec x dx = \int 1 dx = x$$

.

There is no need to include constants of integration, since these just lead to homogeneous solutions in y_p.

$\Rightarrow$

$$y_p = \ln|\cos x|\cos x + x\sin x.$$

Hence we get finally that

$$y = y_h + y_p = c_1\cos x + c_2\sin x + \ln|\cos x|\cos x + x\sin x$$

Example This example is a video slide show. Slide Example

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