Ma 530

Series Solution Near a Singular Point

Finding a Second Linearly Independent Solution

Recall: Near a regular singular point we have
Theorem. If the DE differential equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{1}$$

has a regular singular point at $x = 0$, then there is at least one solution which possesses an expansion of the form

$$y = x^m \sum_{n=0}^{\infty} a_n x^n.$$

where m satisfies the indicial equation

$$m^2 + (p_0 - 1)m + q_0 = 0 \tag{2}$$

and

$$p_0 = \lim_{x \to 0} xP(x)$$
$$q_0 = \lim_{x \to 0} x^2 Q(x)$$

Remark: The motivation for seeking a solution of the form

$$y = x^m \sum_{n=0}^{\infty} a_n x^n$$

of

$$y'' + P(x)y' + q(x)y = 0$$

when $x = 0$ is a regular singular point comes from Euler's equation. Recall the solutions of

$$y'' + \frac{p}{x}y' + \frac{q}{x^2}y = 0$$

where p and q are constants were

$$x^{m_1} \text{ and } x^{m_2}$$

if $m_1 \neq m_2$ where real roots and

$$x^{m_1} \text{ and } x^{m_1} \ln x$$

for a real, repeated root. The most general equation with a regular singular point at $x = 0$ is obtained from Euler's equation by replacing p and q in Euler's equation by power series. Thus we have

$$y'' + \left(\frac{\sum_{n=0}^{\infty} p_n x^n}{x} \right) y' + \left(\frac{\sum_{n=0}^{\infty} q_n x^n}{x^2} \right) y = 0 \qquad (*)$$

Since the transition from Euler's equation to this last equation is accomplished by replacing constants by power series, it is natural to expect or guess solutions of $(*)$ of the form

$$y = x^m \sum_{n=0}^{\infty} a_n x^n$$

or perhaps

$$y = \left[x^m \sum_{n=0}^{\infty} a_n x^n \right] \ln x$$

We will now investigate the nature of the solutions of (1) in more detail than we did before with the idea of gaining insight into how one finds two linearly independent solutions of (1). Again

$$y = x^m \sum_{n=0}^{\infty} a_n x^m = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$y' = \sum_{n=0}^{\infty} a_n (m+n) a_n x^{m+n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (m+n)(m+n-1) x^{m+n-2}$$

The fact that $xP(x)$ and $x^2 Q(x)$ are analytic at $x = 0$ implies that

$$xP(x) = \sum_{n=0}^{\infty} p_n x^n$$

$$x^2 Q(x) = \sum_{n=0}^{\infty} q_n x^n$$

Therefore

$$\begin{aligned} Py' &= \left(\frac{1}{x}\right) \left(\sum_{n=0}^{\infty} p_n x^n \right) \left(\sum_{n=0}^{\infty} a_n (m+n) a_n x^{m+n-1} \right) \\ &= x^{m-2} \left(\sum_{n=0}^{\infty} p_n x^n \right) \left(\sum_{n=0}^{\infty} a_n (m+n) a_n x^n \right) \\ &= x^{m-2} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} p_{n-k} a_k (m+k) \right) x^n \\ &= x^{m-2} \sum_{n=0}^{\infty} \left(\sum_{k=o}^{n-1} p_{n-k} a_k (m+k) + p_0 a_n (m+n) \right) x^n \end{aligned}$$

Also

$$
\begin{aligned}
Q(x)y &= \frac{1}{x^2}\left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^{m+n}\right) \\
&= x^{m-2}\left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
&= x^{m-2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} q_{n-k} a_k\right)x^n \\
&= x^{m-2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n-1} q_{n-k} a_k + q_0 a_n\right)x^n
\end{aligned}
$$

Substituting these expressions for y'', Py', and Qy into (1) and cancelling the factor x^{m-2} yields

$$\sum_{n=0}^{\infty}\left\{a_n[(m+n)(m+n-1)+(m+n)p_0+q_0]+\sum_{k=0}^{n-1} a_k[(m+k)p_{n-k}+q_{n-k}]\right\}x^n = 0$$

Equating the coefficient of x^n to zero yields

$$a_n[(m+n)(m+n-1)+(m+n)p_0+q_0]+\sum_{k=0}^{n-1} a_k[(m+k)p_{n-k}+q_{n-k}] = 0, \quad n = 0,1,2,\ldots \tag{3}$$

Equation (3) is called the recursion formula. Now $n = 0 \Rightarrow$

$$a_0[m(m-1)+mp_0+q_0] = 0$$

which is equation (2) as expected.

Setting $n = 1$ yields

$$a_1[(m+1)(m)+(m+1)p_0+q_0]+a_0(mp_1+q_1) = 0$$

Setting $n = 2$ yields

$$a_2[(m+2)(m+1)+(m+2)p_0]+a_0(mp_2+q_2)+a_1[(m+1)p_1+q_1] = 0$$

In general we have

$$a_n[(m+n)(m+n-1)+(m+n)p_0+q_0]+a_0(mp_n+q_n)+\cdots+a_{n-1}[(m+n-1)p_1+q_1] = 0$$

Let

$$f(m) = m(m-1)+mp_0+q_0$$

Then the above equations may be rewritten as

$$a_0 f(m) = 0$$

$$a_1 f(m+1)+a_0(mp_1+q_1) = 0$$

$$a_2 f(m+2)+a_0(mp_2+q_2)+a_1[(m+1)p_1+q_1] = 0$$

$$a_m f(m+n)+a_0(mp_n+q_n)+\cdots+a_{n-1}[(m+n-1)p_1+q_1] = 0 \tag{$*$}$$

Once m has been determined, then the following equations give a_1 in terms of a_0, a_2 in terms of a_1 and

a_2, etc. The $a_n's$ are therefore determined in terms of a_0 for *each choice of* m *unless*

$$f(m+n) = 0$$

for some positive integer n, in which case the process breaks off. Thus if $m_1 = m_2 + n$ for $n \geq 1$, the choice $m = m_1$ will give a solution, but in general the choice $m = m_2$ does not since $f(m_2 + n) = f(m_1) = 0$. Note also that if $m_1 = m_2$, we have only one solution.

In all other cases where m_1 and m_2 are real numbers the procedure yields two independent formal solutions. (We shall not treat the case when m is complex.) The above may be summarized as:

Theorem A. Assume $x = 0$ is a regular singular point of the differential equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{1}$$

and that the power series expansions of $xP(x)$ and $x^2Q(x)$ are valid on an interval $|x| < R$ $(R > 0)$. Let the indicial equation

$$m^2 + (p_0 - 1)m + q_0 = 0 \tag{2}$$

have real roots m_1 and m_2 with $m_1 \leq m_2$. Then equation (1) has at least one solution

$$y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n \qquad (a_0 \neq 0)$$

on the interval $0 < x < R$ where the a_n's are determined in terms of a_0 by the recursion formula $(*)$ above with m replaced by m_1, and the series $\sum a_n x^n$ converges for $|x| < R$. Furthermore, if $m_1 - m_2$ is not zero or a positive integer, then equation (1) has a second independent solution

$$y_2 = x^{m_2} \sum_{n=0}^{\infty} b_n x^n \qquad (b_0 \neq 0)$$

on the same interval, where in this case the b_n's are determined in terms of b_0 by $(*)$ with m replaced by m_2 and again y_2 converges for $|x| < R$.

Theorem A fails to answer the question of what happens when $m_1 - m_2$ is zero or a positive integer. In order to convey an idea of the possibilities, we distinguish 3 cases.

Case A. If $m_1 = m_2$, there cannot exist a second Frobenius solution.

The other two cases case, in which $m_1 - m_2$ is a positive integer, will be easier to grasp if we insert $m = m_2$ into the recursion formula $(*)$ and write it as

$$a_m f(m_2 + n) = -a_0(m_2 p_n + q_n) - \cdots - a_{n-1}[(m_2 + n - 1)p_1 + q_1] \tag{4}$$

The difficulty arises when $f(m_2 + n) = 0$ for some integer n.

Case B. If the right hand side of (4) is not zero when $f(m_2 + n) = 0$, then there is no possible way of continuing the calculation. There cannot exist a second Frobenius solution.

Case C. If the right hand side of (4) happens to be zero when $f(m_2 + n) = 0$, then a_n is unrestricted and can be assigned any value whatever. In particular, we can put $a_n = 0$ and continue to compute the coefficients without any further difficulties. Hence, in this case there does exist a second Frobenius series solution.

The following will enable us to discover what form the second solution takes when is zero or a positive integer. We begin by defining a positive integer k by

$$k = m_1 - m_2 + 1$$

The indicial equation

$$m^2 + (p_0 - 1)m + q_0 = 0 \tag{2}$$

may be written as

$$(m - m_1)(m - m_2) = m^2 - (m_1 + m_2)m + m_1 m_2 = 0$$

Thus

$$p_0 - 1 = -(m_1 + m_2)$$

or

$$m_2 = 1 - p_0 - m_1$$

and therefore $k = 2m_1 + p_0$. Since

$$y = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$$

is one solution, we seek a second solution by letting

$$y_2 = v(x) y_1$$

Differentiating and substituting into the D.E. (1) $\Rightarrow$

$$v(y_1'' + P(x)y_1' + Q(x)y_1) + v''y_1 + v'(2y_1' + Py_1) = 0$$

or

$$\frac{v''}{v'} = -2\frac{y_1'}{y_1} - P$$

Hence

$$\ln v' = -2\ln y_1 - \int P dx$$

$\Rightarrow$

$$v' = \frac{1}{y_1^2} e^{-\int P dx}$$

so

$$v = \int\left(\frac{1}{y_1^2} e^{-\int P dx} dx\right)$$

Therefore

$$v' = \frac{1}{x^{2m_1}[\sum a_n x^n]^2} e^{-\int\left(\frac{p_0}{x} + p_1 + \cdots\right)dx}$$

since $xP(x) = \sum p_n x^n$. Therefore

$$v' = \frac{1}{x^{2m_1}\left[\sum a_n x^n\right]^2} e^{-p_0 \ln x - p_1 x - \cdots} = \frac{1}{x^k\left[\sum a_n x^n\right]^2} e^{-p_1 x - \cdots} = \frac{1}{x^k} g(x)$$

where $g(x)$ is a function that is analytic near $x = 0$ and

$$g(0) = \frac{1}{a_0^2}$$

so

$$g(x) = \sum_{n=0}^{\infty} b_n x^n$$

Therefore

$$v' = b_0 x^{-k} + b_1 x^{-k+1} + \cdots + b_{k-1} x^{-1} + b_k + b_{k+1} x + \cdots$$

and finally

$$v = b_0 \frac{x^{-k+1}}{-k+1} + b_1 \frac{x^{-k+2}}{-k+2} + \cdots + b_{k-1} \ln x + b_k x + \cdots$$

Thus

$$\begin{aligned} y = v y_1 &= y_1 \left[b_0 \frac{x^{-k+1}}{-k+1} + b_1 \frac{x^{-k+2}}{-k+2} + \cdots + b_{k-1} \ln x + b_k x + \cdots \right] \\ &= y_1 b_{k-1} \ln x + \sum_{n=0}^{\infty} a_k x^{n+m_1} \left\{ b_0 \frac{x^{-k+1}}{-k+1} + b_1 \frac{x^{-k+2}}{-k+2} + \cdots + b_k x + \cdots \right\} \\ &= b_{k-1} y_1 \ln x + x^{m_1 - k + 1} \sum_{n=0}^{\infty} c_n x^n \\ &= b_{k-1} y_1 \ln x + x^{m_2} \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

since $k = m_1 - m_2 + 1 \Rightarrow m_2 = m_1 - k + 1$.

This last expression is then the form of our second solution. It yields some information. First, if m_1 and m_2 are equal, then $k = 1$ and $b_{k-1} = b_0 \neq 0$. In this case–Case A–the term containing $\ln x$ is definitely part of the second solution. However, if $m_1 - m_2 = k - 1$ is a positive integer, then sometimes $b_{k-1} \neq 0$ and the log term is present (Case B) and sometimes $b_{k-1} = 0$ and there is no log term (Case C). [Note that we have no real way of knowing this, since the coefficients of are not readily determined.] In any case when the method of Frobenius fails we know the second solution is of the form

$$y_2 = y_1 \ln x + x^{m_2} \sum_{n=0}^{\infty} c_n x^n$$