Ma 530 Series Solution of DEs

Solution by Power Series

We shall now study ways of solving the second order differential equation

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

This equation has variable coefficients. In any interval where $a_2(x) \neq 0$, we can divide the equation by $a_2(x)$ to obtain $y'' + P(x)y' + Q(x)y = R(x)$. We shall consider only the homogeneous case

$$y'' + P(x)y' + Q(x)y = 0. \quad (1)$$

This equation will be solved by power series. It will turn out that near a point $x = a$

$$y = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots$$
$$= \sum_{n=0}^{\infty} a_n(x-a)^n$$

where $a_0, a_1, \ldots, a_n, \ldots$ are constants to be determined. This series is the Taylor series expansion of the solution y.
Let us first begin with two definitions.

Definition 1. A function $f(x)$ is said to be analytic at $x = a$ if it can be expanded in a power series, in powers of $x - a$, which converges to $f(x)$ in an open interval containing $x = a$. This series is the Taylor series for $f(x)$.

Note: A necessary condition for $f(x)$ to be analytic is that $f(x)$ and its derivatives of all orders exist at $x = a$.
$f(x)$ analytic $\Rightarrow$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This series is called the Taylor series of $f(x)$ near $x = a$.
When the point $x = a = 0$, the series is called MacLaurin series. If $f(x)$ is not analytic at $x = a$, it is said to be singular or to have a singularity at $x = a$.

Examples:
1. $f(x) = \frac{1}{1-x} = (1-x)^{-1}$ is analytic at $x = 0$
$f'(x) = -(1-x)^{-2}(-1) = (1-x)^2$
$f''(x) = +2(1+x)^{-3} \quad f'''(x) = 3.2(1-x)^{-4}$
$f^{(n)}(x) = n!(1-x)^{-n-1}$ so that at $x = 0$ $f^{(n)}(0) = n!$

$\Rightarrow$ Taylor expansion for $\frac{1}{1-x}$ near $x = 0$ is
$f(x) = \sum_{n=0}^{\infty} \frac{n!}{n!}(x-0)^n = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$.
However, $\frac{1}{1-x}$ is not analytic at $x = 1$, since it approaches ∞ as $x \to 1$. $\Rightarrow$ no power series in powers of $(x-1)$.

2. $f(x) = x^{\frac{1}{n}} \quad n = 2, \ldots$ is not analytic at $x = 0$
$f'(x) = \frac{1}{n} x^{\frac{1}{n}-1} \quad n \geq 2 \Rightarrow \frac{1}{n} - 1 < 0 \Rightarrow f'(x)$ at $x = 0$ does not exist.

3. $f(x) = \frac{1}{x^2+1}$ analytic for all real x. However, for x complex, $x = \pm i$ is a singularity.

4. What are the singularities of $f(x) = \dfrac{x-1}{x^3 - 2x^2 + x}$?

$$f(x) = \frac{x-1}{x(x^2-2x+1)} = \frac{x-1}{x(x-1)^2} = \frac{1}{x(x-1)}$$

$\Rightarrow x = 0$ and $x = 1$ are singularities.

Definition 2. The point $x = a$ is called an ordinary point of the differential equation

$$y'' + P(x)y' + Q(x)y = 0 \quad (1)$$

if both $P(x)$ and $Q(x)$ are analytic at $x = a$. If either $P(x)$ or $Q(x)$ is not analytic at $x = a$, then this point is called a singular point or singularity of the differential equation (1).

Example

$$(x^2 - 3x + 2)y'' + \sqrt{x}\,y' + x^2 y = 0$$

We rewrite the equation as

$$y'' + \frac{\sqrt{x}\,y'}{(x-2)(x-1)} + \frac{x^2}{(x-2)(x-1)}y = 0$$

Thus $P(x) = \frac{\sqrt{x}}{(x-2)(x-1)}$ and $Q(x) = \frac{x^2}{(x-2)(x-1)}$. Thus the equation has singularities at $x = 2,\ 1,$ and 0. $x = 0$ is a singular point because the derivative of $P(x)$ at 0 is not defined. All other points are ordinary points.

The theorem below gives conditions which insure the existence of a power series solution.

Theorem. If $x = a$ is an ordinary point of the differential equation (1), then $\exists$ two linearly independent power-series solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x-a)^n$$

These solutions will be valid in some interval containing $x = a$.

Method of Solution Near an Ordinary Point.

Example Consider the differential equation

$$y'' + xy' + 2y = 0.$$

Here $P(x) = x$ and $Q(x) = 2$. They are both analytic $\forall\, x$, and in particular at $x = 0$. Hence by the above theorem $\exists$ two solutions of form

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

(Here $a = 0$) The coefficients a_n are determined from the differential equation as follows.
Now

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

.
The differential equation $\Rightarrow$

$$\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} + x \cdot \sum_{n=1}^{\infty} a_n n x^{n-1} + 2 \cdot \sum_{n=0}^{\infty} a_n x^n = 0.$$

or

$$2a_0 + \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n(n+2)\ x^n = 0.\ (*)$$

We shall combine the coefficients of like powers of x in $(*)$ to get one power series. To do this we must put each term in the equation in the same form. This is accomplished by "shifting" the second series in $(*)$.

If we let $n = k - 2$ in the second series, $(*)$ becomes

$$2a_0 + \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} + \sum_{k=3}^{\infty} a_{k-2}(k)x^{k-2} = 0.$$

Since n and k are "dummy" place keepers, we may replace them by m. Doing this yields

$$2a_0 + 2 \cdot 1 \cdot a_2 + \sum_{m=3}^{\infty} [a_m(m)(m-1) + a_{m-2}(m)]x^{m-2} = 0.$$

Remark. If $\sum_0^{\infty} a_n(x-a)^n = 0 \quad \forall\, x$ in some interval $\Rightarrow a_n = 0$ for $n = 0, 1, 2, ...$
Thus we have from the above equation

1. $2(a_2 + a_0) = 0$ or $a_2 = -a_0$
2. $m = 3 \Rightarrow (3 \cdot 2a_3 + 3a_1) = 0$ or $2a_3 = -a_1 \Rightarrow a_3 = -\frac{1}{2}a_1$
3. $m = 4 \Rightarrow (4 \cdot 3a_4 + 4a_2) = 0$ or $a_4 = -\frac{1}{3}a_2 = +\frac{1}{3}a_0$
4. $\Rightarrow [m(m-1)a_m + ma_{m-2}] = 0$ or $a_m = -\frac{1}{m-1}a_{m-2}$ for $m \geq 3$.

The expression in 4 is called the recurrence relation. Continuing we have for $m = 5$ and $m = 6$
$a_5 = -\frac{1}{4}a_3 = -\frac{1}{4}(-\frac{1}{2}a_1) = \frac{1}{4 \cdot 2}a_1$ and $a_6 = -\frac{1}{5}a_4 = -\frac{1}{5}(\frac{1}{3}a_0) = -\frac{1}{5 \cdot 3}a_0$
Hence the solution is

$$y = a_0 + a_1 + a_2x^2 + \cdots = a_0[1 - x^2 + \frac{1}{3}x^4 - \frac{1}{5 \cdot 3}x^6 + \cdots] + a_1[x - \frac{1}{2}x^3 + \frac{1}{4 \cdot 2}x^5 - \cdots].$$

It can be shown that in general

$$y = a_0[\sum_{n=0}^{\infty}(-1)^n \frac{1}{3 \cdot 5 \cdots (2n-1)}x^{2n} + \ldots] + a_1[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n-1}}{2 \cdot 4 \cdots (2n-2)} + \ldots].$$

The above is the general solution of differential equation with two arbitrary constants a_0 and a_1.
Question: Where is the series solution valid? We shall use the ratio test to determine where the series converges. Recall that if $\lim_{n\to\infty} \left|\frac{b_{n+k}}{b_n}\right| = L$ and $L < 1 \Rightarrow \sum b_n$ converges.
Recall that we have $a_m = -\frac{1}{m-1}a_{m-2}$.

$$\Rightarrow \lim_{m\to\infty} \left|\frac{a_{m+2}x^{m+2}}{a_m x^m}\right| = \lim_{m\to\infty} \left|\frac{-\frac{1}{m+1}a_m x^2}{a_m}\right| = \lim_{m\to\infty}\left|\frac{1}{m+1}x^2\right| = 0$$

$\Rightarrow$ the series converges $\forall\, x$.

In general we have the following result about the convergence of a series solution.

Theorem. If $x = a$ is an ordinary point for the differential equation

$$y'' + P(x)y' + Q(x)y = 0,$$

then $\exists$ 2 L.I. series solutions of the form

$$y(x) = \sum_{0}^{\infty} a_n(x-a)^n.$$

These series converge at least $\forall$ values of x such that $|\ x - a\ | < R$, where R is the distance from the point $x = a$ to the nearest singular point of the D.E. in the complex plane.

Remark. The distance between $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ is
$|\ z_1 - z_2\ | = [(a_1 - a_2)^2 + (b_1 - b_2)^2]^{\frac{1}{2}}$.

Example $(x^2 - 3x + 2)y'' + \sqrt{x}\ y' + x^2 y = 0$
$\Rightarrow$

$$y'' + \frac{\sqrt{x}}{(x-2)(x-1)}y' + \frac{x^2}{(x-2)(x-1)}y = 0$$

$x = 2,\ 1$ are singular points. Also $x = 0$ is a singular point due to the $\sqrt{x}$. $\exists$ a solution of form

$$y = \sum a_n(x-10)^n$$

about 10. By the theorem this converges $\forall x$ such that $|\ x - 10\ | < R$. Since $x = 2$ is the nearest singularity to $x = 10$, $R = |\ 10 - 2\ | = 8$.

Example Find the general solution near $x = 0$ of

$$y'' + xy = 0.$$

$y = \sum_0^\infty a_n x^n$ $y' = \sum_0^\infty n a_n x^{n-1} = \sum_1^\infty n a_n x^{n-1}$ and $y'' = \sum_1^\infty n(n-1)a_n x^{n-2} = \sum_2^\infty n(n-1)a_n x^{n-2}$.

The differential equation $\Rightarrow$

$$\sum_2^\infty a_n(n)(n-1)x^{n-2} + \sum_0^\infty a_n x^{n+1} = 0.$$

We must line up like powers of x. To do this both series must be of the same form. Consider

$$\sum_2^\infty a_n(n)(n-1)x^{n-2} = \sum_{k=-1}^\infty (k+3)(k+2)a_{k+3}x^{k+1}$$

where we have let $n - 2 = k + 1 \Rightarrow n = k + 3$ or $k = n - 3$. When $n = 2 \Rightarrow k = -1$. The D.E. may now be written as

$$\sum_{k=-1}^\infty (k+3)(k+2)a_{k+3}x^{k+1} + \sum_0^\infty a_k x^{k+1} = 0$$

$\Rightarrow$

$$2(1)a_2 + \sum_0^\infty \{(k+3)(k+2)a_{k+3} + a_k\}x^{k+1} = 0$$

$\Rightarrow a_2 = 0$ and

$$a_{k+3} = \frac{-a_k}{(k+3)(k+2)} \qquad k = 0,1,2,\ldots$$

$k = 0 \quad \Rightarrow a_3 = \frac{-a_0}{3 \cdot 2} = \frac{-a_0}{6}$

$k = 1 \quad \Rightarrow a_4 = \frac{-a_1}{4 \cdot 3} = \frac{-a_1}{12} \qquad k = 2 \quad \Rightarrow a_5 = \frac{-a_2}{5 \cdot 4} = 0$

$k = 3 \quad \Rightarrow a_6 = \frac{-a_3}{6 \cdot 5} = \frac{a_0}{30 \cdot 6} = \frac{a_0}{180}$

$k = 4 \quad \Rightarrow a_7 = \frac{-a_4}{7 \cdot 6} = \frac{a_1}{7 \cdot 6 \cdot 12}$

$k = 5 \quad \Rightarrow a_8 = 0$

$k = 6 \quad \Rightarrow a_9 = \frac{-a_6}{9 \cdot 8} = \frac{-a_0}{9 \cdot 8(180)}$ etc.

Hence

$$y = \sum_{k=0}^\infty a_k x^k = a_0[1 - \frac{1}{6}x^3 + \frac{1}{560}x^6 - \frac{1}{72 \cdot 180}x^9 + \cdots] + a_1[x - \frac{1}{12}x^4 + \frac{1}{7 \cdot 6 \cdot 12}x^7 + \cdots].$$

Example This example is a video slide show. Slide Example

You will need Real Player to view this. To get it click on Real Player.

Solution Near a Singular Point

Consider now the case where we seek the solution of

$$y'' + P(x)y' + Q(x)y = 0 \quad (1)$$

near a singular point of either P or Q , i.e. a point where either P or Q are not analytic. We shall use the Method of Frobenius. We cannot treat all singularities. We begin with a definition.

Definition. A point $x = a$ is said to be a regular singular point or a regular singularity of the D.E. (1) if

1. $x = a$ is a singular point of (1); and
2. $(x-a)P(x)$ and $(x-a)^2Q(x)$ are analytic at $x = a$.

Remark:

If $f(x)$ analytic at $x = a \Rightarrow$

$$f(x) = \sum_0^\infty a_n(x-a)^n$$

If $f(x)$ is not analytic at $x = a \Rightarrow$

$$f(x) = \sum_{-\infty}^\infty a_n(x-a)^n = \cdots + \frac{a_{-3}}{(x-a)^3} + \frac{a_{-2}}{(x-a)^2} + \frac{a_{-1}}{(x-a)} + a_0 + a_1(x-a) + \cdots$$

Thus the conditions $(x-a)P(x)$ and $(x-a)^2Q(x)$ are analytic at $x = a$ restrict the amount of singularity that $P(x)$ and $Q(x)$ can have.

Remark. Condition 2 $\Rightarrow$ $(x-a)P(x)$ and $(x-a)^2Q(x)$ have Taylor series at $x = a$. If $x = a$ is a singular point which is not regular, it is called an irregular singular point.

Ex. (1) $x^2y'' + pxy' + qy = 0$ Euler's equation. This may be rewritten as

$$y'' + \frac{p}{x}y' + \frac{q}{x^2}y = 0$$

$x = 0$ is a regular singular point since $xP(x) = x\frac{p}{x} = p$ and $x^2Q(x) = x^2\frac{q}{x^2} = q$

(2)

$$y'' + \frac{2}{x}y' + \frac{3}{x(x-1)^3}y = 0$$

It is clear that $x = 0$ and $x = 1$ are singular points. We must examine each singularity separately to see if it is regular or irregular. Consider $x = 0$ first. Now $xP(x) = 2$ which is analytic near $x = 0$. also $x^2Q(x) = \frac{3x}{(x-1)^3}$ which is also analytic near $x = 0$. Therefore $x = 0$ is a regular singular point.

Now consider $x = 1$. Then $a = 1$ and

$$(x-1)P(x) = \frac{2(x-1)}{x}$$

which is analytic at $x = 1$.

$$(x-1)^2 Q(x) = \frac{3}{x(x-1)}$$

which is *not* analytic at $x = 1. \Rightarrow x = 1$ is an irregular singular point.
Note that we must treat each singular point individually.
Near a regular singular point we have
Theorem. At a regular singular point $x = a$ of the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

there is at least one solution which possesses an expansion of the form

$$y = (x-a)^{\alpha} \sum_{n=0}^{\infty} a_n (x-a)^n.$$

In order to see how one solves equation (1) near a regular singular point $x = a$ in the easiest manner we shall assume $a = 0$. If $a \neq 0$, then let $t = x - a$ in the D.E. and solve in terms of t. $t = 0$ is then a regular singular point.
Now

$$y = x^{\alpha} \sum_{0}^{\infty} a_n x^n = \sum_{0}^{\infty} a_n x^{n+\alpha}$$

$\Rightarrow$

$$y' = \sum_{0}^{\infty} (n+\alpha) a_n x^{n+\alpha-1}$$

and

$$y'' = \sum_{0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha-2}$$

Now $xP(x)$ and $x^2 Q(x)$ are analytic at $x = 0 \Rightarrow$ that

$$xP(x) = \sum_{0}^{\infty} p_n x^n$$

and

$$x^2 Q(x) = \sum_{0}^{\infty} q_n x^n$$

The D.E. $y'' + P(x)y' + Q(x)y = 0$ may be multiplied by x^2 to get

$$x^2 y'' + x^2 P(x) y' + x^2 Q(x) y = 0$$

$$\Rightarrow x^2[\alpha(\alpha-1)a_0 x^{\alpha-2} + (\alpha+1)(\alpha)a_1 x^{\alpha-1} + \cdots]$$
$$+ x[p_0 + p_1 x + \cdots][\alpha a_0 x^{\alpha-1} + (\alpha+1)a_1 x^{\alpha} + \cdots]$$
$$+ [q_0 + q_1 x + \cdots][a_0 x^{\alpha} + a_1 x^{\alpha+1} + \cdots] = 0$$
$$\Rightarrow [\alpha(\alpha-1)a_0 x^{\alpha} + (\alpha+1)\alpha a_1 x^{\alpha+1} + \cdots]$$
$$+ [\alpha p_0 a_0 x^{\alpha} + \alpha p_1 x^{\alpha+1} a_0 + p_0(\alpha+1)a_1 x^{\alpha+1} + \cdots]$$
$$+ [q_0 a_0 x^{\alpha} + q_0 a_1 x^{\alpha+1} + q_1 a_0 x^{\alpha+1} + \cdots] = 0$$

Setting the coefficients of x^{α} equal to $0 \Rightarrow$ $\alpha(\alpha-1)a_0 + \alpha p_0 a_0 + q_0 a_0 = 0$

$\Rightarrow\ \alpha(\alpha-1)+\alpha p_0+q_0=0$ or

$$\alpha^2+(p_0-1)\alpha+q_0=0 \qquad (2)$$

Equation (2) is called the *indicial* equation. This result is not surprising in light of the results we got for Euler's equation. Therefore if α is a root of (2) $\Rightarrow y=\sum a_n x^{\alpha+n}$ is a solution of (1) for this α. The $a_n's$ are determined from the D.E.

Remarks: Since $xP(x)=\sum p_n x^n$ and $x^2Q(x)=\sum q_n x^n$, p_0 and q_o are the first terms in the Taylor expansions of $xP(x)$ and $x^2Q(x)$. Thus

$$p_0=\lim_{x\to 0} xP(x) \text{ and } q_0=\lim_{x\to 0} x^2P(x)$$

Ex. Find a series solution of the D.E.

$$9x^2y''+(x+2)y=0$$

near $x=0$

We rewrite the equation as

$$y''+\frac{(x+2)}{9x^2}y=0$$

$P(x)=0\quad Q(x)=\frac{(x+2)}{9x^2}$ so $x=0$ is regular singular point.

$xP(x)=0=\sum p_n x^n$ so $p_n=0\Rightarrow p_0=0$

$x^2Q(x)=\frac{x+2}{9}=\frac{2}{9}+\frac{x}{9}=\sum q_n x^n\Rightarrow q_0=\lim_{x\to 0}\left(\frac{2}{9}+\frac{x}{9}\right)=\frac{2}{9}$

Therefore equation (2) for α becomes

$$\alpha^2-\alpha+\frac{2}{9}=0$$

or

$$(\alpha-\frac{2}{3})(\alpha-\frac{1}{3})=0$$

and therefore $\alpha=\frac{2}{3}$ or $\alpha=\frac{1}{3}$.

$\Rightarrow$ solutions of the form $y=x^{\frac{1}{3}}\sum_0^\infty a_n x^n$ and $y=x^{\frac{2}{3}}\sum_0^\infty b_n x^n$.

Consider the case $\alpha=\frac{1}{3}$ Since

$$y=\sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$

$$y'=\sum_{n=0}^{\infty}(n+\frac{1}{3})a_n x^{n-\frac{2}{3}} \quad\text{and}\quad y''=\sum_{n=0}^{\infty}(n+\frac{1}{3})(n-\frac{2}{3})a_n x^{n-\frac{5}{3}}$$

D.E. $9x^2\,y''+(x+2)y=0\ \Rightarrow$

$$9\sum_{n=0}^{\infty}\left(n+\frac{1}{3}\right)\left(n-\frac{2}{3}\right)a_n x^{n+\frac{1}{3}}+x\sum_{n=0}^{\infty}a_n x^{n+\frac{1}{3}}+2\sum_{n=0}^{\infty}a_n x^{n+\frac{1}{3}}=0$$

or

$$\sum_{n=0}^{\infty}\left\{9\left(n+\frac{1}{3}\right)\left(n-\frac{2}{3}\right)a_n+2a_n\right\}x^{n+\frac{1}{3}}+\sum_{n=0}^{\infty}a_n x^{n+\frac{4}{3}}=0$$

$\Rightarrow$

$$\sum_{0}^{\infty}\{[(3n+1)(3n-2)+2]a_n\}x^{n+\frac{1}{3}}+\sum_{k=1}^{\infty}a_{k-1}x^{k+\frac{1}{3}}=0$$

Let $k+\frac{1}{3}=n+\frac{4}{3}\Rightarrow k=n+1\Rightarrow$

$$\sum_{0}^{\infty}\{[9n^2-3n-2+2]a_n\}x^{n+\frac{1}{3}}+\sum_{k=1}^{\infty}a_{k-1}x^{k+\frac{1}{3}}=0.$$

Or

$$\sum_{1}^{\infty}\{3m(3m-1)a_m+a_{m-1}\}x^{m+\frac{1}{3}}=0$$

$\Rightarrow$

$$a_m=\frac{-a_{m-1}}{3m(3m-1)}$$

$m=1\Rightarrow a_1=\frac{a_0}{3\cdot 2}\quad m=2\Rightarrow a_2=-\frac{a_1}{6\cdot 5}=+\frac{a_0}{6\cdot 5\cdot 3\cdot 2}=\frac{a_0}{2\cdot 3\cdot 5\cdot 6}$

$m=3\Rightarrow a_3=-\frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}$

Therefore one solution is $y_1=a_0x^{\frac{1}{3}}\left(1-\frac{x}{3\cdot 2}+\frac{x^2}{2\cdot 3\cdot 5\cdot 6}-\frac{x^3}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}+\cdots\right)$

For $\alpha=\frac{2}{3}$ one gets

$y_2=b_0x^{\frac{2}{3}}\left(1-\frac{x}{3\cdot 4}+\frac{x^2}{3\cdot 4\cdot 6\cdot 7}-\frac{x^3}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}+\cdots\right)$

For the method of Frobenius we have

Theorem. If the differential equation

$$y''+P(x)y'+Q(x)y=0$$

has a regular singularity at $x=0$ and if the roots α_1 and α_2 of the indicial equation are distinct and do not differ by an integer, then there are two linearly independent solutions of the form

$$y_1(x)=x^{\alpha_1}\sum_{0}^{\infty}a_nx^n \text{ and } y_2(x)=x^{\alpha_2}\sum_{0}^{\infty}b_nx^n$$

Bessel's Equation

The equation

$$x^2y''+xy'+(x^2-p^2)y=0\quad(1)$$

is known as Bessel's equation. Here p is constant. We shall assume $p\geq 0$. The solutions are called Bessel functions. Now (1) $\Rightarrow$

$$y''+\frac{1}{x}y'+\left(1-\frac{p^2}{x^2}y\right)=0$$

$\Rightarrow$ we have a regular singular point at $x=0$.

Therefore we assume $y=\sum_0^\infty a_nx^{n+\alpha}$

Now $xP(x)=x(\frac{1}{x})=1\Rightarrow p_0=1$ and $x^2Q(x)=x^2\,(1-\frac{p^2}{x^2})=x^2-p^2\Rightarrow q_0=-p^2$.

Therefore, the indicial equation, $\alpha^2+(p_0-1)\alpha+q_0=0$, in this case is

$\Rightarrow \quad \alpha^2 - p^2 = 0 \quad \text{or } \alpha = \pm p$
Consider $\alpha = p \geq 0 \Rightarrow$
$y = \sum_0^\infty a_n x^{n+p} \quad y' = \sum_0^\infty (n+p) a_n x^{n+p-1} \quad y'' = \sum_0^\infty (n+p)(n+p-1) a_n x^{n+p-2}$
D.E. (1) $\Rightarrow$

$$\sum_0^\infty (n+p)(n+p-1) a_n x^{n+p} + \sum_0^\infty (n+p) a_n x^{n+p} + \sum_0^\infty a_n x^{n+p+2} - p^2 \sum_0^\infty a_n x^{n+p} = 0$$

$\Rightarrow$

$$\sum_0^\infty \{(n+p)(n+p-1) + (n+p) - p^2\} a_n x^{n+p} + \sum_0^\infty a_n x^{n+p+2} = 0$$

$\Rightarrow$

$$\sum_0^\infty \{(n+p)^2 - (n+p) + (n+p) - p^2\} a_n x^{n+p} + \sum_0^\infty a_n x^{n+p+2} = 0$$

or

$$\sum_1^\infty (n)(n+2p) a_n x^{n+p} + \sum_{k=2}^\infty a_{k-2} x^{k+p} = 0$$

In the last series we have made the substitution $k + p = n + p + 2$ which implies $n = k - 2$. Replacing the "dummy" variables n and k by m leads to
$\Rightarrow$

$$1(1+2p) a_1 x^{1+p} + \sum_2^\infty \{m(m+2p) a_m + a_{m-2}\} x^{m+p} = 0$$

$\Rightarrow (1+2p) a_1 = 0 \quad a_n = \frac{-a_{n-2}}{n(n+2p)} \quad n = 2, 3, \ldots$
$\Rightarrow a_1 = 0 \;$ and $a_2 = \frac{-a_0}{2(2+2p)} = \frac{-a_0}{4(p+2)}$
$a_3 = 0 \;$ and $a_4 = -\frac{1}{4(4+2p)} a_2 = +\frac{a_0}{4^2(p+1)2(p+2)}$
$a_5 = 0 \;$ and $a_6 = -\frac{a_4}{6(6+2p)} = -\frac{a_0}{4^2 \cdot 6 \cdot 2(p+3) \cdot 2(p+1)(p+2)}$
$= \frac{-a_0}{4^3 \cdot 3!(p+1)(p+2)(p+3)}$

In general

$$a_{2k} = (-1)^k \frac{1}{k! 4^k (p+k)(p+k-1) \ldots (p+2)(p+1)} a_0$$

Therefore one solution is

$$y = a_0 \sum_{k=0}^\infty (-1)^k \frac{x^{p+2k}}{k! 4^k (p+k)(p+k-1) \cdots (p+2)(p+1)}$$

By a proper choice of a_0 we can write y as the conventional Bessel function. Let

$$a_0 = \frac{1}{2^p \Gamma(p+1)}$$

where Γ is the gamma function defined by

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$$

For $s = n$ an integer we may show that $\Gamma(n) = (n-1)!$
Also $\Gamma(s) = (s-1)\Gamma(s-1) \quad s > 1$

$$\Rightarrow$$

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \, x^{p+2k}}{k!\{2^{2k}2^p\}\{\Gamma(p+1)(p+k)\ldots(p+1)\}}$$

$$= \sum_0^\infty \frac{(1-1)^k}{k!\Gamma(p+k+1)} \left(\frac{x}{2}\right)^{p+2k}$$

This is called a Bessel Function of the first kind of order p.

It may be shown J_p converges $\forall x \geq 0$ if $p \geq 0$. For the case $p = n$ an integer $\Gamma(p+k+1) = (n+k)!$.

$$\Rightarrow$$

$$J_n(x) = \sum_0^\infty \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k}$$

Remark. There are many relations between Bessel functions. For example,

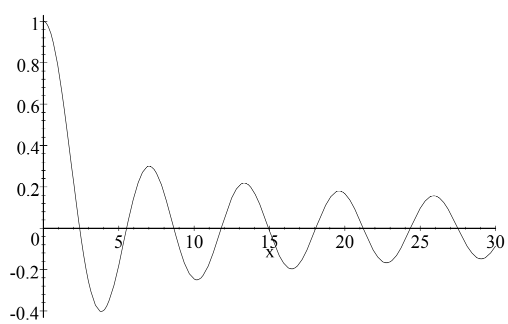
$$J_0(x) = \sum_0^\infty \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}$$

$\Rightarrow$

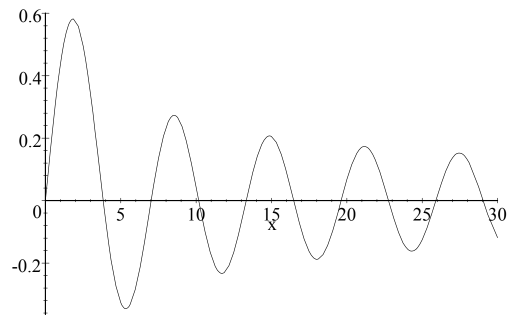
$$J_0'(x) = \sum_1^\infty (2k) \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k-1} \left(\frac{1}{2}\right) = \sum_1^\infty \frac{(-1)^k}{k!(k-1)!} \left(\frac{x}{2}\right)^{2k-1}$$

$$= \sum_0^\infty \frac{(-1)^{n+1}}{(n+1)!n!} \left(\frac{x}{2}\right)^{2n+1} = -J_1(x)$$

The graph of $J_0(x)$ and $J_1(x)$ are given below.



$J_0(x)$



$J_1(x)$